Different Compositions of Animal Spirits and Their Impact on Macroeconomic Stability

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Abstract

Several economic surveys specify an aggregate sentiment as the difference between optimists and pessimists. This concept is also applied in agent-based modelling where, governed by endogenously determined transition probabilities, the individual agents switch between two attitudes. The present paper extends this stylized framework by adding a third category, which may be viewed as neutrality. On this basis it then puts forward a dynamic three-dimensional Goodwinian model with a special focus on multiple long-run equilibrium positions, which may emerge from just one and very natural non-linearity in the switching process. The equilibria exhibit the same difference between optimists and pessimists and thus give rise to the same aggregate rate of growth, so that they cannot be distinguished at the macroeconomic level. The feature in which they nevertheless differ is the share of neutral agents. Remarkably, this affects stability. In particular, the trajectories may converge to one of two locally stable equilibrium points, or alternatively to a uniquely determined limit cycle. Coexistence of these attractors is absent in a two-state sentiment dynamics.

JEL classification: C 13, E 12, E 30.

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1 Introduction

A fundamental issue in which heterodox macroeconomic theory differs from the orthodoxy is the notion of expectations, where it determinedly abjures the Rational Expectations Hypothesis. Instead, to emphasize its view of a constantly changing world with its fundamental uncertainty, heterodox economists frequently refer to the famous idea of the animal spirits. This is not only a useful keyword for general conceptual discussions, in the last decade there have also been several attempts to incorporate the notion into a rigorous modelling of macroeconomic dynamics. A recent overview of this literature is given in Franke and Westerhoff (2017).

This work even provides a certain microfoundation of macroeconomic behaviour, albeit, of course, a rather stylized one. Striving for a canonical unifying framework, it proved very convenient to refer to a large population of agents who face a binary decision. Most prominently, they may choose between optimism and pessimism, or between two types of expectations about prices or demand. Individual agents do this with certain probabilities and then take a decision. The central point is that these probabilities endogenously change in the course of time, adjusting upward or downward in reaction to the agents' observations. The latter may include output, prices as well as an 'average opinion' or an 'aggregate sentiment', which represents the aforesaid animal spirits. As a consequence, agents constantly switch between two attitudes or two strategies. Their decisions vary correspondingly, as does the macroeconomic outcome resulting from them.

By the law of large numbers, this can all be cast in terms of aggregate variables, where one such variable represents the current population mix, i.e. the animal spirits. The relationships between the variables form an ordinary and well-defined macrodynamic system specified in discrete or continuous time, as the case may be. The variations of the aggregate sentiment play a crucial role and the system's dynamic properties are basically determined by the mechanism that governs the switching of the agents. In particular, a possible and often attractive result could then be recurrent waves of optimism and pessimism.

The choice between two attitudes is not only a most elementary device. It also has a direct counterpart in several economic surveys when they determine an aggregate sentiment of their respondents. Questions are here of the kind, how do you assess the future development of an economic variable or the economic situation in general: better, worse, or about equal? Reported to the public is nevertheless a single indicator, which derives from the difference between optimists and pessimists and neglects the number of intermediate answers. Thus, the majority index that is typically considered in the economic models with their binary choices stands in a one-to-one correspondence to this statistic.

Despite the three possible attitudes in these surveys, there is a continuum of distributions where the number of neutral respondents varies but the difference between the respondents with the polar answers remains fixed. In line with the surveys' focus on their indicator variable, this suggests that also from a theoretical point of view the neutral attitude is of secondary importance. To give an example, consider a model with a population of firms that, with respect to their decisions about fixed investment, are either optimistic, pessimistic, or neutral. Correspondingly, they increase their capital stock at a high, a low, and a medium rate of growth, where the latter is just the mean value of the other two. If furthermore the size distributions of the capital stocks do not systematically differ between the three groups, the average capital growth rate will be the same for all configurations with the same difference between the optimistic and pessimistic investment decisions. From a macroeconomic perspective it therefore seems that the particular share of the neutral agents plays no essential role.

It is here where the present paper intervenes. Its aim is to demonstrate that such a conclusion is not the whole story when we widen the horizon and study this situation in a dynamic setting. It will turn out that the binary decision modelling may in fact neglect vital information. An extension to a three-state sentiment modelling is not only closer to human perception, it may also generate dynamic phenomena that were not possible in the simpler framework.

For our investigations, we start out from the canonical transition probability approach as it was introduced into economics by the seminal work of Kirman (1993) and Lux (1995).[1] The abovementioned probabilities are here the likelihood of an individual optimist to become a pessimist and *vice versa*. This approach gained attraction because of the pleasant nonlinearity it implies and also because it can capture contagion effects, or the phenomenon of herding, the strength of which can be conveniently measured by a so-called herding coefficient.[2] While maintaining the concept of the transition probabilities and their specification, in this paper we will allow the agents to switch across all three attitudes.

In principle, this opens up a variety of new effects, which could be regulated by quite a number of additional parameters in the model. For our present purpose, however, we resist the temptation of unfolding the full richness of mutual influences of the three attitudes on each other. To begin with, we rather model the switching process in a most parsimonious way, such that finally there will again be only one behavioural parameter to consider. Thus concentrating on the herding effects alone, a pure sentiment dynamics is obtained, which can be studied as a deterministic two-dimensional differential equation system in the shares of the optimists and pes-

[1] For a succinct presentation of its potential, see Franke (2014) or Franke and Westerhoff (2017).

[2] The coefficient governs the functional relationship between the probabilities and the model's majority index, i.e. the difference between the shares of optimists and pessimists.

simists. Interestingly, depending on the herding intensity, this basic system can already have up to seven equilibrium points.

In a next stage of the analysis, we include a constant term in the probability functions and study how parametric shifts of it affect the previous dynamics in the phase plane. This will help us to understand what is going on when, in a third stage, we integrate the sentiment dynamics into a macroeconomic context. Concretely, as in the example sketched above, we consider a population of firms facing an investment decision. *Via* the multiplier, their aggregate outcome determines the output of the entire economy, which in turn, through a Phillips curve relationship, determines the changes in real wages. On the other hand, we introduce an elementary feedback from the income distribution between workers and capitalists on the firms' investment activities. Together this leads to a system with the wage share as a third dynamic variable. It can be viewed as another development of Goodwin's (1967) seminal growth cycle model, which puts special emphasis on the sentiment dynamics that are underlying the firms' fixed investment.

Again depending on the herding parameter, this system can (generically) have one or three steady state positions, in which the economy grows at constant proportions. In case of multiple equilibria, all three of them exhibit the same rate of growth, they only differ in the share of firms with a neutral attitude and thus a medium capital growth rate. Nonetheless, some of these equilibria are locally stable and others unstable. Starting near an unstable equilibrium, the economy may converge to a stable growth path, but there are also good chances that it eventually enters a periodic motion. In the latter case, the trajectories display the predator-prey characteristics of the famous Goodwin cycles of income distribution and economic activity.

The point that we want to make is thus the following. We can have two configurations that are identical from a macroeconomic point of view and where also the difference between optimistic and pessimistic firms is the same. They only differ in the share of neutral firms. As it will be found, the position with the higher share of neutral investors is conducive to stability, the other equilibria can but need not necessarily be stable. The role of the neutral agents is therefore by no means negligible. More generally, as stylized as it is, agent-based modelling and the precise composition of an aggregate sentiment formed by heterogeneous attitudes or strategies can also be relevant for macroeconomic discussions. In short, the composition of 'animal spirits' may matter for stability.

The remainder of the paper falls into six sections. The next four sections prepare the ground for our little macroeconomic model. Section 2 presents the transition probability framework for the sentiment adjustments. Section 3 sets up a pure herding dynamics where the population shares only feed back on themselves, i.e. they react to the current majorities. This leads to two differential equations in the shares of the bold and cautious agents, which are conveniently analyzed by considering their isoclines in the phase plane. Section 4 studies the number of equilibria that may possibly occur, their stability properties, and the basins of attraction of the stable rest points. Section 5 introduces an exogenous term in the feedback relationships and investigates how parametric shifts of it affect the number and location of the equilibria. The macroeconomic side is then added to this model in Section 6. The populations shares determine investment and thus, *via* a multiplier, economic activity, while the previously constant term is interpreted as the wage share and endogenized *via* a Phillips curve relationship. The resulting three-dimensional system constitutes our Goodwinian agent-based model, for which we will subsequently demonstrate the dynamic properties that have just been pointed out. Section 7 concludes.

2 A three-state transition probability approach

Let us consider a population of agents who in each point in time can be either optimistic, pessimistic, or indifferent. For a more suggestive notation, we will also call these agents bold, cautious, and neutral and use the symbols b, c and n as superscripts to identify the corresponding attitudes. Likewise, as pure characters the symbols denote the population shares of these agents, where certainly $n = 1 - b - c$.

The agents change their attitudes in the course of time. This process is governed by transition probabilities. Accordingly, let π^{rs} be the probability for an agent with attitude r to switch to attitude s $(r, s = b, c, n)$. More exactly, these probabilities are uniform across group r, and they are the probabilities *per unit of time*. The latter means that $\Delta t\, \pi^{rs}$ is the agent's probability for switching within an adjustment period of length Δt. As below we will let Δt shrink to zero, π^{rs} need not necessarily be less than unity. The probabilities are predetermined and fixed within a given adjustment period.

Our first simplification in the interest of parsimony is the assumption that agents cannot switch directly from bold to cautious or contrariwise, i.e. $\pi^{bc} = \pi^{cb} \equiv 0$. Instead, a bold agent first has to turn neutral before he can become cautious. This intermediate step is not very restrictive if two successive probabilities like π^{bn} and π^{nc} are relatively large. As a consequence, there are only two probabilities determining the change in the share of bold agents between t and $t + \Delta t$, and two probabilities for the change in the share of the cautious agents. Working with an infinitely large population to let the random effects disappear (the intrinsic noise, so to speak), the inflow and outflow for the bold attitude gives us

$$b_{t+\Delta t} \;=\; b_t \,+\, n_t\, \Delta t\, \pi^{nb}_t \,-\, b_t\, \Delta t\, \pi^{bn}_t$$

Similarly so for the share of cautious agents. Going to the limit as regards the adjustment period, $\Delta t \to 0$, and expressing the share of neutral agents by the share of the polar attitudes, a deterministic formulation in continuous time is obtained:

$$\begin{aligned} \dot{b} &= (1-b-c)\,\pi^{nb} \;-\; b\,\pi^{bn} \\ \dot{c} &= (1-b-c)\,\pi^{nc} \;-\; c\,\pi^{cn} \end{aligned} \tag{1}$$

The pair of the population shares is contained in the simplex $S_{bc} := \{(b,c) \in [0,1]^2 : b+c \leq 1\}$, which is a triangle in the (b,c)-plane. Of course, the further significance of (1), hinges on the specification of the transition probabilities. It will, in particular, guarantee that the shares (b,c) cannot leave the simplex, so that (1) will indeed remain well-defined.

To describe the variations of a transition probability, we conceive it as a function of an index variable. This variable, which from now on will be called a switching index, is a real function of, in principle, any dynamic variable(s) in the model that the agents observe. The use of the switching index, on the other hand, is standardized and carried over from the framework with two attitudes (Franke, 2014; Franke and Westerhoff, 2017). We introduce an index s_b for the probabilistic flux of agents towards and out of a bold attitude, and analogously an index s_c for the cautious agents. The indices can attain positive and negative values. As a convention, let rising values of s_b increase the transition probability π^{nb} and decrease the probability π^{bn} in the opposite direction. Likewise for the switching index s_c.

Regarding the functional relationship between the probabilities and the corresponding switching index, we follow the device introduced by Weidlich and Haag (1983) for the switching between two (arbitrary) attitudes, strategies, etc. It is accordingly assumed that the *relative* changes of π^{nb} and π^{bn} in response to the changes in s_b are linear and symmetrical, that is, $d\pi^{nb}/\pi^{nb} = A\,ds_b$ for some constant A and $d\pi^{bn}/\pi^{bn} = -A\,ds_b$. As a consequence, the function of the transition probabilities is proportional to the exponential function $\exp(s_b)$.[3] The treatment of π^{nc} and π^{cn} is the same. Together, the following functional specification of the four transition probabilities towards and out of the two extreme positions is adopted,

$$\begin{aligned} \pi^{nb} = \pi^{nb}(s_b) \;&=\; \mu\,\exp(s_b) \qquad\qquad & \pi^{bn} = \pi^{bn}(s_b) \;&=\; \mu\,\exp(-s_b) \\ \pi^{nc} = \pi^{nc}(s_c) \;&=\; \mu\,\exp(s_c) & \pi^{cn} = \pi^{cn}(s_c) \;&=\; \mu\,\exp(-s_c) \end{aligned} \tag{2}$$

The exponential function in (2) will be the only essential nonlinearity in our modelling work. Note that it results from a very natural assumption; any other specification would appear more arbitrary or at least require more discussion. In particular,

[3] As it is only a matter of scaling the switching index, the constant A can be taken to be unity without loss of generality.

the switching indices s_b, s_c themselves will later be linear functions. All the more remarkable will be the rich effects that the exponential function alone can generate.

Technically speaking, μ is a positive integration constant. With a given unit of time, it can also be interpreted as a parameter that measures how fast agents react to variations in the switching index. Weidlich and Haag (1983, p. 41) therefore call ν a flexibility parameter. Certainly, the exponential functions ensure strictly positive values of the probabilities. The complementary condition that they are bounded, which they are if the switching indices remain bounded, should be a property of the model into which (2) is incorporated. It may also be noticed that a possible risk aversion, according to which the agents show a greater tendency to switch to the neutral rather than one of the extreme attitudes, could be captured by a suitable specification of the switching indices.

Using (2), the two differential equations (1) for the evolution of the population shares b and c become

$$\begin{aligned} \dot{b} &= F_b(b,c,s_b) := \mu\,[(1-b-c)\exp(s_b) - b\exp(-s_b)] \\ \dot{c} &= F_c(b,c,s_c) := \mu\,[(1-b-c)\exp(s_c) - c\exp(-s_c)] \end{aligned} \tag{3}$$

An important conceptual observation is that in a state of rest of system (3), where not only $\dot{b} = \dot{c} = 0$ but additionally $s_b = s_c = 0$, there is still a number of individual agents who are randomly switching from one attitude to another, owing to $\pi^{nb} = \pi^{bn} = \pi^{nc} = \pi^{cn} = \mu > 0$ for their transition probabilities. That is, there is always motion at the micro level, even though these changes may balance at the macro level.

Provided the switching indices remain bounded, it is also immediately seen that the pairs (b,c) cannot leave the simplex S_{bc}. Moreover, they are even repelled from its boundary. In fact, when the fraction of bold agents b tends toward unity and therefore that of the cautious agents c to zero, the second term in the square bracket of the first equation of (3) will eventually dominate the first one, so that the derivative of b will become negative then. The same reasoning for the second equation prevents $c \to 1$. On the other hand, when b and c both tend to zero, i.e. all agents tend to become neutral, the first term in the bracket of either equation will eventually dominate the second one, so that eventually $\dot{b} > 0$, $\dot{c} > 0$ and $\dot{n} < 0$.

While eq. (2) provides a first and useful organizational device, the meaningfulness of the modelling rests on the variables that make up the switching indices. In the next stage of the analysis it is expedient to concentrate on how the agents react to each other—or infect each other— without any reference to their macroeconomic environment and how it evolves with their actions. Less technically, agents switch towards an attitude just because others have already done it before. This is what is commonly described as herding. The corresponding system will be called a pure sentiment dynamics.

3 The pure sentiment dynamics

We may imagine individual agents in the pure sentiment dynamics as asking their friends, business partners, competitors, etc., for their opinion about the future, whereupon they may show a certain tendency to join the majority. Following the crowd is not necessarily an unreasonable behaviour; especially persons who have no special knowledge of the circumstances may think that other people are better informed than they. The channels of information spreading may also be more indirect, through the mass media or more specialized sources from which the agents can derive information about the distribution of current attitudes. In any case, the basic notion is that a widely disseminated attitude tends to attract further adherents.

Considering the probability $\pi^{nb}(\pi^{nc})$ of switching from neutral to bold (cautious), this idea says that $\pi^{nb}(\pi^{nc})$ increases when the number of bold (cautious) agents rises relative to the number of neutral agents. It is thus the difference in the shares of the two attitudes that enters the function of the switching indices, $b-n = b-(1-b-c) = 2b+c-1$ and $c-n = 2c+b-1$, respectively. The intensity of this feedback is measured by a uniform coefficient ϕ_h, where the subscript 'h' may be reminiscent of 'herding'; ϕ_h will therefore be referred to as our herding coefficient. Presently neglecting all other possible influences, the two switching indices are given by,[4]

$$\begin{aligned} s_b &= s_b(b,c) &=& \phi_h\,(2b+c-1) \\ s_c &= s_c(b,c) &=& \phi_h\,(2c+b-1) \end{aligned} \tag{4}$$

In addition, the agents may have a certain predisposition towards one of the attitudes. This could be captured by adding or subtracting a constant term in s_b and/or s_c. The resulting effects are studied in the next section, though with a view to a different interpretation of the term.

In sum, the pure sentiment dynamics that will be the basis for the later macroeconomic extension is set up by the equations (3) and (4). As it has already been emphasized and will be seen shortly, the nonlinearity given by the exponential function is by no means negligible. Depending on the herding coefficient ϕ_h, it can give rise to more than one and even more than three equilibrium points.[5] This feature can be conveniently studied by considering the isoclines $\dot{b}=0$ and $\dot{c}=0$ and their

[4]Generally, the difference between the polar attitudes might play a role as well. Franke (2018) shows that this introduces a new and interesting feature for the geometry of the isoclines in the phase plane. Again, it is for the sake of simplicity that we abstain from such an additional influence.

[5]As is well-known, a pure sentiment dynamics where the agents can only switch between two attitudes can be reduced to a one-dimensional differential equation in the difference between the two population shares. Here up to three equilibria are possible if the herding coefficient is sufficiently high; see Lux (1995, p. 886).

points of intersection in the phase plane. More precisely, these intersections must occur within the domain of the process, the triangle S_{bc}.

It goes without saying that the flexibility coefficient ν takes no effect on the locus of the isoclines, it only governs the speed of the adjustments. Furthermore, one type of isoclines is also unaffected by variations of the herding parameter ϕ_h. Note, to this end, that $s_b = 0$ if $2b+c-1=0$ and that this implies $\dot{b} = (1-b-c) - b = -2b-c+1 = 0$. Hence $\dot{b}=0$ if a pair (b,c) lies on the straight line $2b+c-1 = 0$. By the same token, $\dot{c}=0$ if a pair (b,c) lies on the straight line $2c+b-1=0$. Let us denote these linear isoclines by LI_b and LI_c, respectively. Evidently, they intersect at $(b^s,c^s) := (1/3,1/3)$. That is, independently of ϕ_h, one equilibrium point of (3), (4) is always constituted by the fully symmetric distribution of the attitudes, where $b=c=n=1/3$.

The herding coefficient has, however, a bearing on the stability of the symmetric equilibrium. The Jacobian matrix evaluated at this point reads,[6]

$$J = J(b^s,c^s) \quad = \quad \begin{bmatrix} -2+4\phi_h/3 & -1+2\phi_h/3 \\ -1+2\phi_h/3 & -2+4\phi_h/3 \end{bmatrix}$$

As this is a symmetric matrix, it has real eigen-values, which rules out cyclical behaviour around this point. The determinant of the matrix is always positive, whereas the trace is negative for low values of ϕ_h and positive for high values. In detail, the symmetric equilibrium (b^s,c^s) is a locally stable node if $\phi_h < 3/2$, and an unstable node if $\phi_h > 3/2$.

The points on the linear isoclines LI_b need not be the only points (b,c) that entail $\dot{b}=0$. The geometric locus of the second isocline can, however, no longer be explicitly computed. That is, for a given value of b, conditions can be analytically established for the existence of solutions to the implicit equation $F_b[b,c,s(b,c)]=0$ in c that add to the already known pair $(b,c) \in LI_b$. The implicit function theorem guarantees that there is a differentiable function $c=c(b)$ with $F_b[b,c,s(b,c)]=0$ for all b in a certain interval; and although it has no closed-form representation, one can compute its derivative dc/db. This allows one to reconstruct the function $c=c(b)$. The analytical effort can nevertheless be saved because the herding coefficient ϕ_h is here the only parameter involved. Thus, for each value of ϕ_h it suffices to calculate the function $c=c(b)$ by numerical methods (i.e. by using an iteration procedure). The geometric outcome is presented in Figure 1.

The shaded triangle is the domain S_{bc} of the sentiment process. Note first that the increasing dashed line is the locus of equal shares of bold and cautious agents, $b=c$. We call it the symmetry line, partly for this reason, but also because the

[6]The computation of this and the other Jacobian matrices later on, together with their eigenvalues, can be obtained from the author on request.

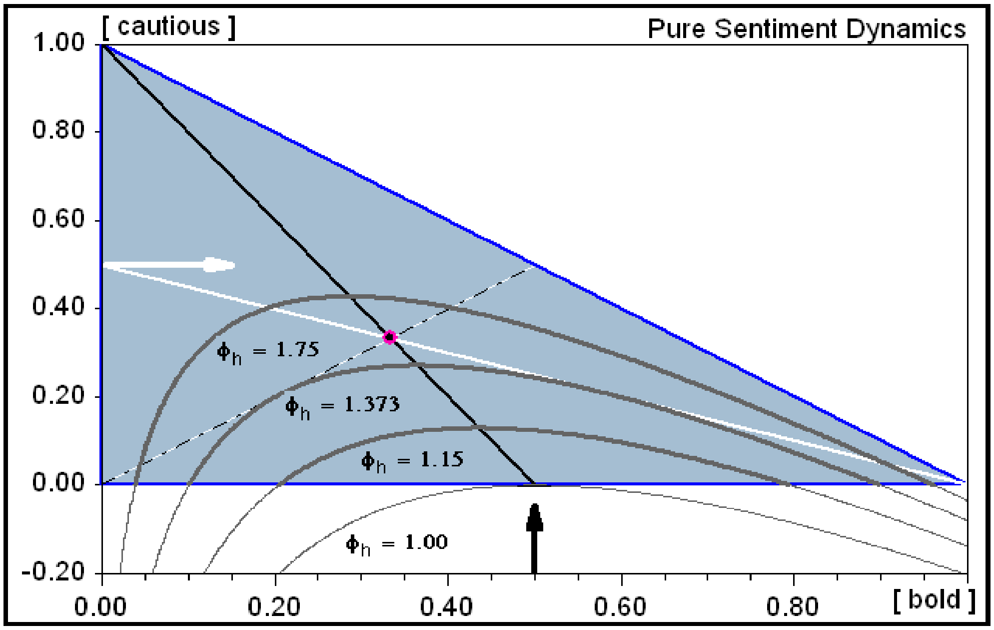

Figure 1: The parabolic isoclines PI_b for selected values of ϕ_h.

Note: The shaded triangle is the simplex S_{bc}. The dashed line is the symmetry line on which $b = c$, the black straight line is the isocline LI_b, the white one is LI_c. The black arrow indicates that, as ϕ_h increases, the parabolic isoclines $PI_b(\phi_h)$ rise upwards from the point $(b,c) = (0.50, 0.00)$ into the simplex. The horizontal white arrow indicates that their counterparts $PI_c(\phi_h)$ grow symmetrically from left to right into the triangle from the point $(b,c) = (0.00, 0.50)$.

isoclines $\dot{c} = 0$ are symmetric to the isoclines $\dot{b} = 0$ with this line as their symmetry axis. The black downward sloping straight line is LI_b, while its counterpart LI_c is drawn as the white straight line. The (red) point at their intersection is the fully symmetric equilibrium $(b^s, c^s) = (1/3, 1/3)$. However, the diagram receives its main interest from the curved black lines, which are the additional isoclines $\dot{b} = 0$ for four selected values of ϕ_h. Because of their parabolic shape, we denote them by $PI_b = PI_b(\phi_h)$. The bold black arrow indicates that they reach more and more into the simplex as ϕ_h increases. The counterparts $PI_c = PI_c(\phi_h)$ for $\dot{c} = 0$ are not shown in order not to overload the diagram, but the white arrow indicates that (instead of vertically) they grow horizontally into the simplex with rising values of ϕ_h.

As long as the herding coefficient is less than unity, the parabolic functions are irrelevant since their graphs lie entirely below or to the left of the simplex, respectively. For $\phi_h = 1.00$, $PI_b(\phi_h)$ touches the line $c = 0$ from below at $b = 0.50$, the same point at which the linear isocline LI_b hits the bottom line of the simplex. The parabolic isoclines $PI_b(\phi_h)$ begin to play a role as ϕ_h rises above unity, when a part of their graphs reaches into the simplex. For a while this has nevertheless no major effect on the dynamics; $PI_b(\phi_h)$ still lies below both isoclines for $\dot{c} = 0$, the linear one LI_c and $PI_c(\phi_h)$. Hence there is no additional point of intersection of two isoclines and thus no other equilibrium.

This observation remains true for all coefficients $\phi_h < 1.373$ (rounded). When ϕ_h assumes this critical value, $PI_b(\phi_h)$ reaches so far into the simplex that it touches LI_c, which gives us a second equilibrium point. Symmetrically (not shown), $PI_c(\phi_h)$ touches LI_b and we obtain a third equilibrium. Even more, at the same time both parabolic isoclines $PI_b(\phi_h)$ and $PI_c(\phi_h)$ touch the symmetry line and therefore each other. Hence we have a total of four equilibria that upon a gradual increase of the herding coefficient suddenly come into being at $\phi_h = 1.373$.

At a further slight increase of ϕ_h, the tangential points bifurcate into two points where the corresponding isoclines intersect. This raises the number of equilibria up to seven. Apart from the symmetric equilibrium (b^s, c^s), two lie on the linear isocline LI_b, two on LI_c, and two on the symmetry line. Clearly, both points on LI_b (on LI_c, on the symmetry line) will be between (b^s, c^s) and the lower-right corner (the upper-left corner and the origin, respectively).

Proceeding with the increase of ϕ_h, both parabolic isoclines will eventually go through (b^s, c^s). This happens at $\phi_h = 1.50$. As a consequence, three points of the aforementioned intersections collide into the intersection (b^s, c^s) of the two linear isoclines. That is, at this special value of ϕ_h the number of equilibria drops to four. The old configuration with the seven equilibria is re-established and the three equilibrium points reappear when ϕ_h rises above 1.50. The only difference is now that the latter are on the other side of the symmetric equilibrium. In Figure 1 this is illustrated with the parabolic isocline $LI_b(\phi_h)$ for $\phi_h = 1.75$.

A scenario with all seven equilibria will be shown in the next section, when we study the dynamic properties of the pure sentiment dynamics. It will then, in particular, be seen which of them are stable and how their basins of attraction are separated from each other.

Before turning to this issue, a paper by Foster and Flieth (2002) should be mentioned that put forward a three-state sentiment dynamics already more than fifteen years ago. Unfortunately, it went largely unnoticed in the macroeconomic literature. The parameters it includes are similar to ours: a so-called interaction strength, which corresponds to our herding coefficient, and three predisposition parameters θ_0, θ_1, θ_2 adding up to unity, which correspond to our supposition of no predisposition if $\theta_0 = \theta_1 = \theta_2 = 1/3$. The necessary nonlinearity in the functional specification of the transition probabilities uses a quadratic term instead of our exponential function, which we obtained from a natural behavioural assumption. In this way, Foster and Flieth get up to five equilibrium points as their interaction strength increases. Generally, the derivation of their dynamic equations appeared somewhat unwieldy and less flexible with respect to extensions of the model by additional feedback effects.[7]

4 Multiple stable equilibria and their basins of attraction

Already with a view to the numerical scenario that we will choose for our macroeconomic model in the next section, let us concentrate on the herding coefficient $\phi_h = 1.42$ for a prior analysis of the pure sentiment dynamics (3), (4). Recalling the critical value of this parameter for the stability of the symmetric equilibrium, (b^s, c^s) is still locally stable then. Furthermore, this value gives rise to the full set of seven equilibria. Figure 2 shows the resulting phase plane.[8] As before, the shaded area in the (b,c)-plane is the domain of the process. The dashed line is the symmetry line on which $b = c$, while the bold black and white lines are the isoclines $\dot{b} = 0$ and $\dot{c} = 0$, respectively.

Stability of the equilibria (i.e. of their points of intersection) is indicated by (red) filled dots, empty dots signify instability. Thus, four of the seven equilibria are stable, the remaining three unstable. Without checking the numerics, the type of instability can be readily determined from an elementary mathematical fact: if two isoclines are upward-sloping and one first cuts the other from below and then a second time from above, the determinant of the Jacobian matrices evaluated at these equilibria changes sign. Likewise if both curves are downward-sloping. If, therefore, one point of intersection of the two parabolic isoclines $PI_b(\phi_h)$ and $PI_c(\phi_h)$ is stable (thus having a positive determinant), the other must be a saddle point (since it has a negative determinant). Analogously for the other pairs of consecutive intersections of two isoclines. As a result, all three unstable equilibria exhibit saddle point instability.

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[7] Very recently, Gomes and Sprott (2017), inspired by rumour propagation theory, proposed sentiment adjustments for five confidence levels about the future performance of the economy: neutrality, exuberant optimism, non-exuberant optimism, exuberant pessimism, non-exuberant pessimism. Introducing the hyperbolic tangent as a convenient exogenous nonlinearity (whereas, in comparison, we would characterize our exponential functions as an 'endogenous' nonlinearity), their system (unlike ours) can produce cyclical and even chaotic dynamics. At first sight, the authors' specifications have a different background from ours, but it may be worthwhile to inquire into possible analogies and conceptual relationships, or lack thereof.

[8] Reference in the title to $z = 0.0000$ becomes clear in the next section.

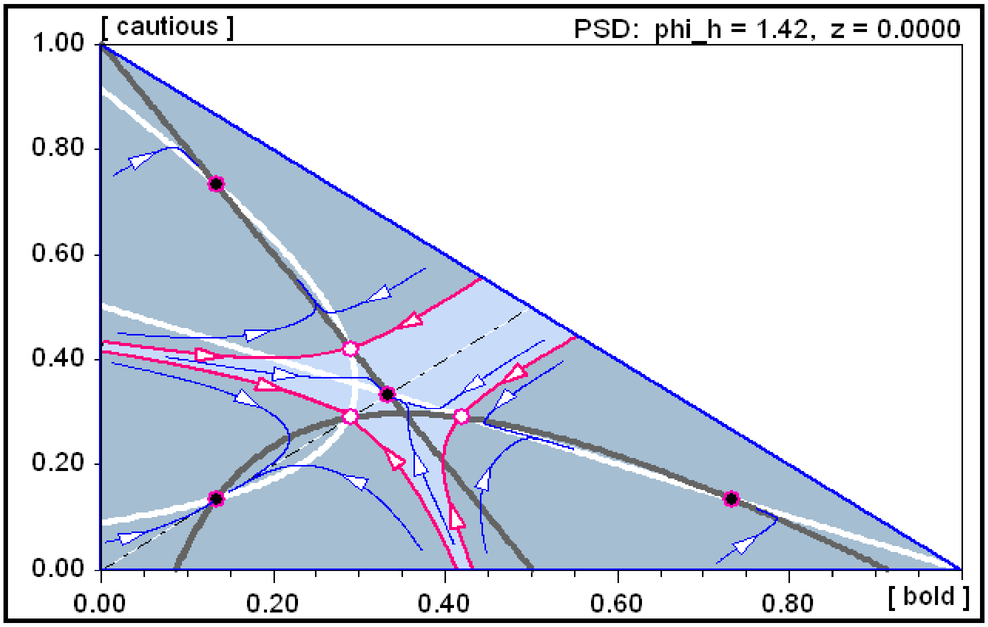

Figure 2: Phase plane of the pure sentiment dynamics (3), (4) under $\phi_h = 1.42$.

Note: The dashed line is the symmetry line on which $b = c$. The bold black (white) lines are the isoclines $\dot{b} = 0$ ($\dot{c} = 0$). Filled (unfilled) red dots indicate stable equilibria (saddle points). Arrows on the thicker (red) lines identify the stable saddle paths, the thin (blue) lines are sample trajectories converging to the different stable equilibrium points.

The thicker (red) lines with an arrow on them depict the stable arms of the three saddle points.[9] They are of particular importance since they separate the basins of attraction of the four stable equilibria. For example, the lighter area which is enclosed by the three pairs of saddle paths shows us the set of all points from which the trajectories converge to the symmetric equilibrium. The region below the saddle paths in the south-west is the basin of attraction of the other stable equilibrium on the symmetry line, where $b = c$ but the neutral agents form a majority. Let us denote an equilibrium of this type by (b^{mn}, c^{mn}). The upper pair of the saddle paths fences off the stable equilibrium (b^{mc}, c^{mc}) in the upper-left corner, where the cautious agents are in the majority. The pair of saddle paths to the right fences off the equilibrium (b^{mb}, c^{mb}) in the lower-right corner, where the agents in their majority herd towards boldness. The thin (blue) lines are sample trajectories and the

[9] They can be easily computed by running the system backwards in time.

arrows on them indicate in which way which equilibrium is approached.

It may finally be mentioned that an increase of the herding coefficient ϕ_h above 1.50 shifts the parabolic isoclines so far into the interior of the simplex that their second point of intersection, i.e. the saddle point on the symmetry line, moves to the right of the symmetric equilibrium. As we already know, the latter is thus destabilized. On the other hand, the stability of (b^{mn}, c^{mn}) in the lower-left corner is maintained. It just shifts towards the origin, that is, its share of neutral agents tends towards 100% as the herding becomes stronger and stronger.

5 Shifts in the switching indices

When introducing the switching indices (4), it was indicated that a possible predisposition of the agents towards the bold or cautious attitude could be reflected by a constant positive or negative term in s_b and s_c. Such a supplement will be considered in the following. Its discussion is, however, organized with a view to a different interpretation in the next section, where the additional term will represent an additional feedback on the switching indices from a macroeconomic variable.

In a full model, the changes in this variable will in turn be influenced by the changes in the agents' population shares b, c and n. Before studying such a higher-dimensional system, it is expedient to remain in the present limited framework and analyze the implications of independent parametric shifts of the additional term. Accordingly, denote the term by z and consider the augmented switching indices

$$\begin{aligned} s_b &= s_b(b,c) &=& \ \phi_h\,(2b+c-1) \,-\, z \\ s_c &= s_c(b,c) &=& \ \phi_h\,(2c+b-1) \,+\, z \end{aligned} \tag{5}$$

The convention that an increase in z weakens rather than strengthens the tendency to switch towards boldness derives from the economic interpretation of z in the next section. Certainly, this weakening should manifest itself in some way in the dynamic properties of the process (3), (5).

For a better comparison of the effects resulting from gradual changes in the parameter z, we maintain the herding coefficient $\phi_h = 1.42$ and reproduce the phase plane of the sentiment dynamics in Figure 2 in the top-left panel (a) in Figure 3. The sample trajectories are now omitted because our first interest is in the shifts of the isoclines and the corresponding points of intersection, where some of them may even disappear as z is sufficiently increased.

A first remarkable and fundamental effect of non-zero values of z is shown in panel (b) in Figure 3. As marginal as the increase of z from zero to $z = 0.0002$ is, both isoclines for $\dot{b} = 0$, $PI_b(\phi_h)$ as well as LI_b, break up near the previous

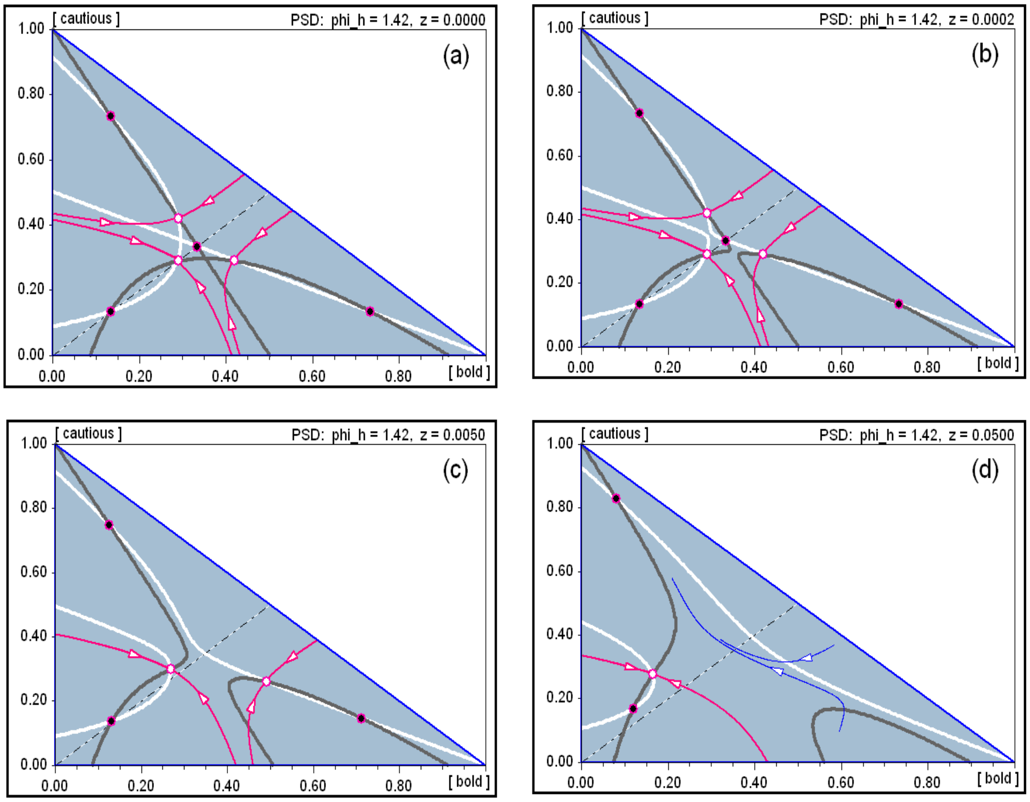

Figure 3: Sentiment dynamics (3), (5) with shifts in the switching index.

Note: The dashed line is the symmetry line on which $b = c$. The bold black (white) lines are the isoclines $\dot{b} = 0$ ($\dot{c} = 0$). Filled (unfilled) red dots indicate stable equilibria (saddle points), arrows indicate stable saddle paths. Underlying is again the herding coefficient $\phi_h = 1.42$.

symmetric equilibrium (b^s, c^s). The same applies to the two isoclines for $\dot{c} = 0$. In this way, the upper part of LI_b (which is perfectly linear no more) connects to the left part of $PI_b(\phi_h)$, and the lower part of LI_b connects to the the right part of $PI_b(\phi_h)$. That is, we now have a left and a right isocline $\dot{b} = 0$, both of which are curved lines. Similarly so for LI_c and $PI_c(\phi_h)$, where we now have a left and a right isocline $\dot{c} = 0$. Despite this qualitative change, the position of their points of intersection is only slightly affected and also their stability properties remain the same.

A further increase in z shifts the right isocline $\dot{c} = 0$ to the right and the left isocline $\dot{b} = 0$ to the left, especially in their middle segments. Although the quantitative effect does not appear to be very dramatic, it is sufficient to dissolve the lower two points of intersection on the right isocline $\dot{c} = 0$, near the previous symmetric equilibrium. Panel (c) illustrates this phenomenon for $z = 0.0050$. The other equilibria and their stability properties are not essentially affected, The disappearance of (the counterpart of) the symmetric equilibrium has nevertheless considerably enlarged the basin of attraction of the stable equilibrium (b^{mc}, c^{mc}) with its large majority of cautious agents. On the other hand, the basin of attraction of the bold equilibrium (b^{mb}, c^{mb}) is only moderately reduced.

Panel (d) in Figure 3 demonstrates what happens if the increase in z proceeds. At $z = 0.0500$, the right isocline $\dot{c} = 0$ has shifted so far to the right that now its two points of intersection with the right isocline $\dot{b} = 0$ dissolve. Only two stable equilibria of type (b^{mc}, c^{mc}) and (b^{mn}, c^{mn}), respectively, are left. Their basins of attraction are separated by the stable paths of the only remaining saddle point, where the basin of the cautious equilibrium (b^{mc}, c^{mc}) covers by far the largest region of the domain of the dynamics.

It will be observed that with the rising values of z the left isocline $\dot{c} = 0$ retreats more and more to the left edge of the simplex. The left isocline $\dot{b} = 0$ does so, too, but to a weaker extent. Eventually, therefore, the left isocline $\dot{c} = 0$ lies entirely to the left of the left isocline $\dot{b} = 0$. That is, their two points of intersection disappear and (b^{mc}, c^{mc}) is the only equilibrium remaining. Because, as has already been mentioned in Section 2, the boundaries of the domain are reflecting, (b^{mc}, c^{mc}) with its strong majority of cautious agents is globally stable then.

In all variants of the sentiment dynamics, convergence towards the equilibria occur in a monotonic manner. The sequence of the phase planes in Figure 3 can, however, give us an idea of how cyclical trajectories could come about if a third variable is introduced into the adjustment process. Suppose the attitudes are related to the investment decisions of firms and aggregate investment is linked to the difference between bold and cautious firms. *Via* the multiplier, a higher difference $(b - c)$ means higher economic activity in general, which in turn improves the position of workers and lets their wages increases. In a simple setting this is tantamount to a rising share of wages in national income. Then, let the latter be represented by our shifting term z.

Now, consider a situation with $z = 0$ in the phase plane below the symmetry line and outside the basin of attraction of the symmetric equilibrium (b^s, c^s), such that $(b - c)$ increases. The correspondingly rising wage share z causes (b^s, c^s) to vanish. Quite likely, (b, c) is already so close to the bold equilibrium (b^{mb}, c^{mb}) in the lower-right corner that it is not attracted by the cautious equilibrium (b^{mc}, c^{mc}), either.[10] Thus, z continues its increase—until eventually the equilibrium (b^{mb}, c^{mb})

[10] Of course, these points are "equilibria" with respect to temporarily frozen values of the shifting variable z.

disappears.

From then on, the sentiment adjustments turn around and (b,c) is heading north-west towards (b^{mc},c^{mc}), as indicated by the two sample trajectories in Figure 3(d). Accordingly, the difference $(b-c)$ begins to decline and sooner or later becomes negative. While the resulting decline in the wage share z will re-establish the bold equilibrium (b^{mb},c^{mb}), (b,c) will already have moved so far towards the upper-left corner that it is not captured by the basin of attraction of (b^{mb},c^{mb}). Hence the mechanism is reversed: with the continuing fall in the wage share it is now the equilibrium (b^{mc},c^{mc}) that will finally disappear, whereupon the corresponding enlargement of the basin of attraction of (b^{mb},c^{mb}) initiates a return of (b,c) into this direction.

This process opens up the possibility of persistent oscillations of (b,c) between the upper-left and lower-right corner. The vanishing and re-appearing equilibria and the resulting turnaround in the movements of some dynamic variable(s) are much reminiscent of the geometric reasoning that was first introduced into the analysis of dynamic economic systems by Kaldor (1940) in his famous trade cycle model.

The next section has to provide the finer conceptual and numerical details of these mechanisms and make sure that the idea works out as anticipated. Moreover, it has to be checked what becomes of the previous equilibria of the two-dimensional sentiment dynamics when this process is augmented by the feedbacks from the changes in the wage share variable; that is, which of the original equilibria continue to exist as points of rest and, among those that do, which of them preserve their (in)stability properties.

6 The macroeconomic model

6.1 A Goodwinian augmentation of the sentiment dynamics

Our aim in this section is, first, to incorporate the above sentiment dynamics into a simple macroeconomic framework determining economic activity, and second, to introduce the wage share as an additional dynamic variable that is driven by this activity but also feeds back on the agents' sentiment. We will provide conditions where these dynamic interactions give rise to persistent cyclical behaviour that is conceptually much in line with the distributional oscillations in the seminal Goodwin (1967) model. On the other hand, it will be seen that this is not the only possible outcome of the model.

To begin with the demand side, we concentrate on firms and their fixed investment as the driving force of the economy. Each individual firm has three investment options. If a firm is neutral, it adopts a growth rate g^o of its capital stock that, for convenience, may be conceived of as the economy's 'natural' rate of growth. A firm adopts a higher capital growth rate g^b if is is currently bold, and a lower growth rate g^c if it is cautious. Assuming there are no systematic shifts in the size distribution of the firms in the three investment categories, the aggregate capital stock grows at the rate

$$g \;=\; b\,g^b \,+\, (1{-}b{-}c)\,g^o \,+\, c\,g^c$$

In order to remain as close as possible to our previous setting, let us suppose that the bold and cautious growth rates are symmetric to the natural growth rate g^o, i.e., $g^b = g^o + \gamma$ and $g^c = g^o - \gamma$ for some $\gamma > 0$. The deviations of the actual capital growth rate from its natural value are then strictly proportional to the difference between bold and cautious firms,

$$g \;=\; g(b,c) \;\;=\;\; g^o \,+\, \gamma(b-c) \tag{6}$$

Although several surveys ask their subjects for an optimistic (bold), pessimistic (cautious) and indifferent (neutral) attitude, what they provide as an aggregation of the answers to their queries is an index derived from the difference between the shares of the optimistic and pessimistic respondents. Correspondingly, in the present context, let us call this difference the aggregate *investment climate*. By virtue of our symmetry assumption, it is all we need to determine aggregate investment. This feature is also the reason why the small-scale agent-based macro models so far were only concerned with the direct switches between optimism and pessimism, as this is particularly true for the models by Franke (2008, 2012) and Lojak (2018). What we are going to demonstrate in the following is that nevertheless this reduction misses out some dynamic phenomena.

The other components of demand are assumed to be proportional either to the capital stock K, which may serve as a trend indicator, or to disposable income, which itself is supposed to be a linear function of total income Y and 'trend' K. We furthermore assume a closed economy and continuous temporary equilibrium on the goods markets, so Y represents total output, too. With suitable constants $c_1, \ldots, c_4$ and σ the economy's (constant) average propensity to save out of disposable income, the market clearing condition gives Y = net investment + capital depreciation + consumption + government spending $= gK + c_1K + (1{-}\sigma)\cdot$disposable income $+ c_2K = gK + (c_1{+}c_2)\,K + (1{-}\sigma)\,(Y - c_3K) = gK + c_4K + (1{-}\sigma)\,Y$. Put $\beta_u = c_4/\sigma$ and let $u = Y/K$ denote the output-capital ratio, which indicates the utilization of the capital stock. Then, dividing the market clearing condition by K and solving it for u, a simple multiplier relationship is obtained: $u = g(b,c)/\sigma + \beta_u$.

Again in the interest of simplicity, 'normal' capital utilization, designated u^n, is supposed to prevail if in a balanced position where $b = c$, fixed investment lets the capital stock grow at the normal rate g^o. Thus $u^n = g^o/\sigma + \beta_u$ and, invoking (6), $u - u^n = [g(b,c) - g^o)/\sigma = \gamma(b-c)/\sigma$, where β_u has dropped out again.[11] It is slightly more convenient to refer to the output gap y as a dimensionless indicator of economic activity, which is the percentage deviation of actual output Y from its trend $Y^\star$. If, with respect to the present equipment K, production at $Y^\star$ ensures normal utilization, i.e. $Y^\star = u^n K$, we have $y = (Y - Y^\star)/Y^\star = (Y/K)/(Y^\star/K) - 1 = (u - u^n)/u^n$. Putting $\beta_{gs} = \gamma/(\sigma u^n)$, the output gap is therefore directly linked to the investment climate,

$$y \;=\; (u - u^n)/u^n \;\;=\;\; \beta_{gs}\,(b - c) \tag{7}$$

The sentiment adjustments determining the changes in b and c are no longer a pure herding dynamics. This self-referential mechanism is now augmented by what could be called a hetero-referential mechanism (Orléan, 1995), which takes account of the more 'objective' factors that may induce firms to change their attitude. These factors may reinforce or put a curb on the herding tendencies, or they may even reverse them. Focussing on the firms' profitability in a Goodwinian spirit, we assume that higher profits in the economy as a whole tend to make firms more optimistic about the future. Correspondingly, they increase the probability of a neutral firm to become become bold or, if it is already bold, to maintain this attitude. Conversely, higher profits decrease the probability of a neutral firm to switch to a cautious attitude.

The overall profitability is measured by the share of aggregate profits in national income. As it is common in a Goodwinian context, we refer to the wage share as its complement. Denote it by v. According to what has just been said, this macro variable should be included as a negative entry in the switching index s_b, and as a positive entry in the switching index s_c. Assuming a uniform responsiveness in this respect, too, the strength of this objective feedback factor is represented by a positive coefficient ϕ_v. Relating the wage share to a level v^o that is regarded as 'normal' by the firms, the two switching indices in (4) are generalized as follows,

$$\begin{aligned} s_b \;&=\; s_b(b,c,v) \;=\; \phi_h\,(2b + c - 1) \;-\; \phi_v\,(v - v^o) \\ s_c \;&=\; s_c(b,c,v) \;=\; \phi_h\,(2c + b - 1) \;+\; \phi_v\,(v - v^o) \end{aligned} \tag{8}$$

It remains to specify a law governing the changes in the wage share. This can be done without much effort by employing a straightforward wage Phillips curve.

[11] The structure of the latter relationship would be preserved if, following a common practice in macroeconomic theory, the utilization gap is added as another determinant of investment in (6).

Letting w be the nominal wage rate, z labour productivity and p the price level, it reads,

$$\hat{w} \;=\; \hat{z} + \hat{p} + \beta_w (1-v)\, y \tag{9}$$

The sum of productivity growth $\hat{z}$ and price inflation $\hat{p}$ in (9) constitutes a reference level for wage inflation. Underlying this interpretation is the fact that in a state of normal utilization $u = u^n$, where the output gap y is zero, the wage share $v = wL/pY = (w/p)/(Y/L) = (w/p)/z$ would remain invariant (L being employment), which it does by virtue of $\hat{v} = \hat{w} - \hat{p} - \hat{z} = 0$. Current wage changes exceed the benchmark $\hat{p} + \hat{z}$ if the capital stock is overutilized. Because of overtime work and the higher demand for labour in general, this is a situation where firms will be more willing to make wage concessions. The slope $\beta_w > 0$ is the key coefficient to measure the strength of these reactions. The additional multiplicative term $1-v$ is a convenient device that in principle could prevent wages from totally exhausting the national product.[12]

The Phillips curve is readily translated into a differential equation for the wage share as our third dynamic variable. Logarithmic differentiation of v yields $\hat{v} = \hat{w} - \hat{p} - \hat{z} = \beta_w (1-v)\, y$ and plugging in the relationship (7) for the output gap, we have

$$\dot{v} \;=\; \beta_w \beta_{gs}\, v\, (1-v)\, (b-c) \tag{10}$$

To sum up, the pure sentiment dynamics (abbreviated PSD) are augmented in a Goodwinian spirits by feedbacks from and to income distribution, which is characterized by the wage share v. As it is specified, a three-dimensional differential equation system in the variables (b, c, v) is obtained, constituted by the equations (3), (8), (10). Let us refer to this system as the Goodwinian sentiment dynamics, GSD. Because of our specific purpose, we do not analyse GSD in its full generality (for which a number of different cases would have to be distinguished), but fix the herding coefficient coefficient ϕ_h at the value from our demonstrations above, $\phi_h = 1.42$. The following investigations proceed from there.

6.2 Dynamic analysis

Neglecting economically meaningless situations with $v = 0$ or $v = 1$, it is obvious from the wage share adjustments in (10) that any stationary point of GSD requires

[12] The conceptual idea is that the position of workers in the wage negotiations becomes somewhat weaker when real wages (deflated by labour productivity) have already risen to a high level (that is, $(w/p)/z = (w/p)/(Y/L) = wL/pY = v$). In our present investigations this issue will, however, play no role and there would only be marginal numerical changes if the term $1-v$ were omitted (and β_w correspondingly decreased).

a balance of bold and cautious firms, $b = c$. Geometrically, the pairs (b,c) must lie on the symmetry line in their unit simplex. Three steady state positions, each with an equilibrium wage share $v = v^o$, can thus be immediately recognized, because $\dot{b} = 0$ and $\dot{c} = 0$ in GSD is then brought about by the same values of (b,c) as in Figure 2 for PSD with $\phi_h = 1.42$; this feature is independent of the new parameters in GSD. On the other hand, there is no additional equilibrium, either. The reason is that then its wage share $\tilde{v}$, say, would be different from v^o in the switching indices s_b, s_c, which in PSD corresponds to a nonzero value of the shift variable z. However, any point of intersection of the two isoclines in PSD with, according to (8), z fixed at $z = \phi_v(\tilde{v} - v^o)$ would be off the symmetry line; cf. Figure 3(c) and (d) for an illustration.

While the location of the equilibria of GSD depend on ϕ_h only, the other parameters may have a bearing on their stability properties. A numerical analysis suffices to make our main points. Let us work with the following benchmark scenario, where the underlying time unit is one year:[13]

$$\mu \;=\; 1.00 \qquad \phi_v \;=\; 40.00 \qquad \beta_{gs} \;=\; 0.043 \qquad \beta_w \;=\; 1.50 \tag{11}$$

A reason for this particular choice will be given shortly. The equilibrium wage share is just a matter of scaling. For concreteness later on, we set it at a familiar order of magnitude, $v^o = 0.70$.

Table 1 reports the stability features of the three steady state positions of GSD, beginning with the numerical values of their population shares of bold and cautious firms. Inspection of the Jacobian matrices in the equilibrium points of PSD show that, because of $\partial\dot{v}/\partial v = 0$, a stable equilibrium under GSD requires its stability under PSD (as the matrices of the two- and three-dimensional dynamics have the same trace). At least numerically, however, stability under PSD does not turn into instability under GSD, either. This is stated in the third column of the table.

The stable equilibrium with a majority of 73% neutral firms is again approached in a monotonic manner, as indicated by its leading eigen-value (abbreviated EV in the table), which is real.[14] Given the existence of such an equilibrium, the unstable equilibrium must intuitively be a saddle point. In fact, besides a positive real eigen-value the latter has a pair of complex eigen-values with negative real parts. Mathematically, they form a two-dimensional stable manifold in the three-dimensional state space of (b,c,v), which bounds the basin of attraction of the stable equilibrium.

Also the fully symmetric equilibrium preserves the local stability from PSD. Now, however, its local dynamics is determined by a pair of complex eigen-values,

[13] This is relevant for an assessment of the values of ν and β_w.

[14] Leading eigen-value means the one with the largest real part; its characteristic solution will eventually dominate those of the other eigen-values.

Composition	Shares $b = c$	Stability	Leading EV	Period
Fully symmetric:	0.333	stable	complex	7.40
Neutral majority:	0.134	stable	real	—
Intermediate:	0.286	saddle	real	—

Table 1: Properties of the three equilibria of GSD.

so that generically convergence is cyclical. The period of the thus generated cycles is 7.40 years. The two thick (blue) sample trajectories in Figure 4, which projects the dynamics of GSD onto the (b,c)-plane, illustrate the dynamic behaviour of the population shares around the two stable equilibria. The white and the dark-gray (green-shaded) areas around them indicate their basins of attractions for trajectories starting at the equilibrium wage share. Perhaps somewhat surprisingly, the basins are almost identical to those in PSD (with the wage share fixed at v^o, so to speak). The differences are hard to see with the naked eye: the stretches of the basin of the symmetric equilibrium near the boundaries $c = 0$ and $b = 0$, which are rather narrow under PSD, are here even somewhat narrower.

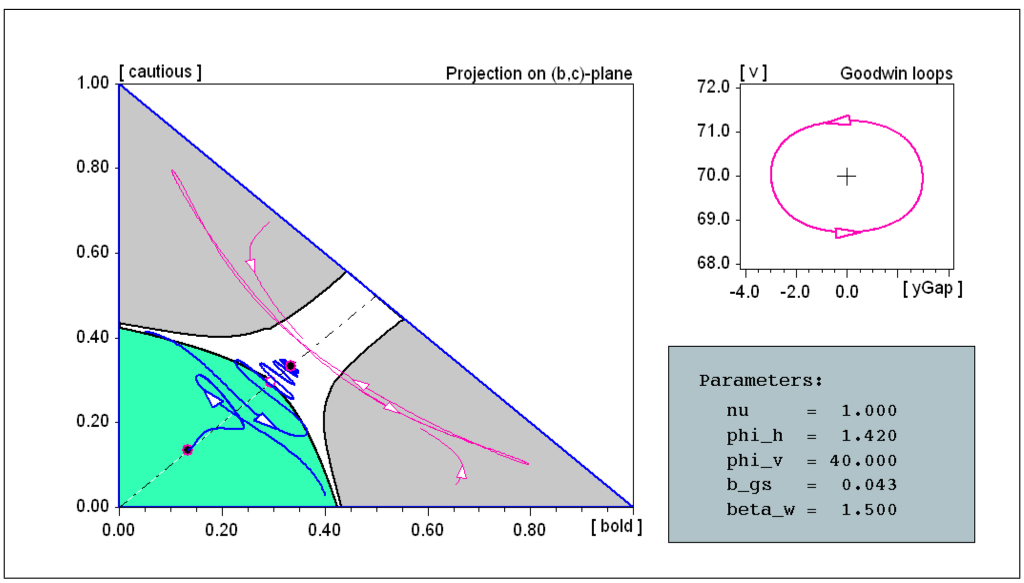

Figure 4: Projection of GSD on the (b,c)- phase plane for the benchmark scenario (11) (for trajectories starting in $v = v^o$).

Remaining are the two regions to the lower-right and upper-left of the (b,c) simplex. For PSD, these are approximately the basins for the trajectories heading towards the stable equilibria in these corners. As the latter are no longer present in GSD and the trajectories starting from within the two gray regions cannot converge to one of the stable equilibria, the dynamics will not cease fluctuating around in the state space.[15] More specifically, it exhibits no chaotic behaviour (which would in principle be possible), but all trajectories starting in the gray regions converge to a limit cycle which is, by all appearances, uniquely determined. This cycle is drawn as the thin (red) closed orbit, so that one can see the slight twist at the extreme values of b and c. The two short trajectories exemplify its attraction.

The small panel in the top-right corner of Figure 4 displays the macroeconomic features of the limit cycle. Plotting the output gap at the horizontal and the wage share at the vertical axis, counterclockwise loops are obtained. This is the manifestation of the well-known Goodwin cycles, and also the economic mechanisms bringing them about are basically the same as in the original Goodwin model and the many models that refined it in some way or another.

The wage share exhibits a phase shift relative to the output gap of approximately a quarter of a cycle. In the second half of an expansion, the workers participate in the favourable economic situation and their share in national income starts rising. This, in turn, leads to the famous profit squeeze, which puts a curb on the average capital growth rate and with it the output gap. The wage share nevertheless continues its increase after the break of the boom, because capacities are still overutilized. Its rise, however, slows down, and roughly in the middle of the contraction the wage share begins to fall. This initiates the second half of the cycle, where the motions are reversed.

The numerical parameters (11) achieve a certain calibration of the growth cycle, where beforehand the flexibility parameter ν is fixed free-hand. To assess our choice of $\mu = 1.00$, recall that without any feedbacks in the switching process, when the switching indices s_b, s_c are zero, the transition probability of an individual firm is $\pi = \mu \exp(0) = \mu > 0$. It can thus be said that on average it is once per year that a firm changes its attitude for idiosyncratic reasons. *A priori*, this seems a reasonable assumption to make. If it is felt that higher or lower values of ν are more appropriate, the following discussion sketches how to proceed.

Given a value of ν, the remaining three coefficients $\phi_v, \beta_{gs}, \beta_w$ are able to bring about three summary statistics that we wish the Goodwin cycle to exhibit. On the one hand, this is a period of $T = 8.00$ years, which is roughly the average length of the US business cycles after 1960. The two other statistics are the amplitudes of

[15] Theoretically, the specification of the Phillips curve prevents the wage share from leaving the unit interval.

the output gap and the wage share. As a straightforward reference we can consult Barboso-Filho and Taylor (2006, p. 390) from the literature on Goodwin cycles. For the US economy they summarize that capacity utilization scaled around a value of one fluctuates by five to seven percentage points over a cycle, and the wage by two or three points. Thus, letting yDev and vDev be the maximal deviations of y and v from their equilibrium values, we aim at yDev = 3.00% and vDev = 1.25%.

Without involving too many digits in setting the parameters, the values in (11) yield $T = 7.98$, yDev = 2.99%, vDev = 1.26%. The model is therefore flexible enough to meet these elementary calibration standards.[16] Table 2 reports how the three summary statistics of the limit cycle change upon *ceteris paribus* variations of the parameters (a hyphen means that the induced changes are hardly noteworthy). Already this limited qualitative information about the mixed reactions shows that there should be sufficient scope for calibrating the model to the numerical statistics that may appear desirable.

	T	yDev	vDev
$\beta_{gs} \uparrow$:	$\downarrow$	$\uparrow$	$\uparrow$
$\phi_v \uparrow$:	$\downarrow$	—	$\downarrow$
$\beta_w \uparrow$:	$\downarrow$	—	$\uparrow$
$\mu \uparrow$:	$\downarrow$	—	$\downarrow$
$\phi_h \uparrow$:	—	$\uparrow$	$\uparrow$

Table 2: The effects of *ceteris paribus* changes of the parameters on the Goodwin limit cycle.

For completeness, the table in addition reports the effects of exogenous changes in the other two parameters, the flexibility ν and the herding coefficient ϕ_h. These reactions may be useful to know for extended versions of the present model,

[16] If the coefficient β_w appears somewhat high, it may be taken into account that on average the Phillips curve (9) has a slope of $(1-\nu^o)\,\beta_w = 0.30 \cdot 1.50$. With additional effects on wage formation, deterministic or stochastic, it is quite likely that similar calibration attempts would yield smaller values for β_w.

when there may also be more summary statistics that one would like the model to match.

The discussion of our model should end with a comparison to the very similar Goodwin model put forward by Franke (2008), the only difference of which is that its firms only switch between optimism and pessimism. In this simpler version there is just one equilibrium point, which is independent of the model's parameters and corresponds to our fully symmetric equilibrium. It is stable for low and unstable for high values of the herding coefficient. In the latter case, it is repelling and all trajectories starting outside this position converge to a unique limit cycle (which can be similarly calibrated as above).

Also in the present case, stronger herding can destabilize the symmetric equilibrium. Again, as in the pure sentiment dynamics, the critical value for this to occur is $\phi_h = 1.50$. By contrast, the equilibrium in the lower-left corner of Figure 4 with its dominating share of neutral firms continues to exist and to be locally stable, at least over a certain range of values of ϕ_h above 1.50. This perhaps something surprising possibility is something that is totally missing in the two-state Goodwin model. We are thus led to conclude that limiting the dynamic modelling of the sentiment adjustments to just two polar states may in fact give away some vital information.

7 Conclusion

The paper started out from the notion of an aggregate sentiment and its modelling with a population of agents who switch between two polar attitudes, most prominently optimism and pessimism, or boldness and caution as we called it. Using the concept of endogenously determined transition probabilities that, in particular, allow for herding tendencies, this approach proved very useful in capturing the idea of 'animal spirits' in macroeconomic dynamics. As an immediate extension of this framework, the paper proposed to include a neutral attitude as a third option for the agents. This does not only strive for higher generality as such. Even if it is specified as parsimoniously as possible, it is able to generate dynamic phenomena that cannot possibly come about with two-state sentiment adjustments. This richness is rooted in just one, very natural nonlinearity in the switching process; no sophisticated combination of a couple of suitably constructed nonlinear mechanisms is required.

Subsequently, the sentiment adjustments more specifically referred to a population of firms where the three attitudes represent different levels of investment. In a Goodwinian spirit, the adjustment equations were augmented with a Phillips curve and repercussions of the wage share on the investment decisions. This led to a three-dimensional macro model. Its analysis concentrated on a scenario with three different long-run equilibria. They exhibit the same difference between bold and cautious firms and thus the same aggregate rate of growth. The only feature where they differ is the share of firms with a neutral attitude. It turns out that this has an effect on the economy's stability: two of the three equilibria are locally stable and the third one is a saddle point. However, the two basins of attraction do not fill out the entire state space. Consequently, trajectories starting outside these regions converge to a periodic motion. This limit cycle displays the familiar Goodwinian features and is unique by all appearances. On the whole we have three co-existing attractors: two stationary points and a closed orbit (a set of points). In finer detail, a high share of neutral agents is favourable for the economy to converge to its steady state growth path.

In our deterministic setting, a trajectory's long-run behaviour is determined once a starting point is given. Because the firms' attitudes have a psychological or sociological component, they may nevertheless be susceptible to incoming news and other exogenous influences. Furthermore, the resulting changes in the population shares may also be relatively large. At any rate, changing, for example, from bold to neutral in order to wait for things to develop is not so dramatic and consequential as changing directly to the opposite attitude. In these cases a sudden regime switching may occur : from persistent fluctuations around the steady state growth path to a cyclical or even monotonic convergence to it, or *vice versa*. This would be one easy explanation for the varying amplitudes in the business cycle as they are observed in reality.

To sum up, the paper has advanced a canonical agent-based sentiment dynamics with three instead of the usual two attitudes between which the agents can switch. It has then integrated this module into a simple macroeconomic context with a strong Goodwinian flavour. This was already sufficient to generate multiple attractors of different types and generally the possibility of regime switching—something that will not be obtained with just two attitudes. Accepting the framework of the transition probability approach, the rest of the modelling was straightforward and unsophisticated. In terms of modelling effort and dynamic phenomena resulting from it, it is a remarkable cost-benefit relationship. There are thus good reasons to expect that combining a three-state sentiment dynamics with other macroeconomic elements will prove similarly fruitful.[17]

[17] It may be noted that the changes in our macroeconomic variable (the wage share) is only dependent on the difference between bold and cautious firms. Additional equilibria not lying on the symmetry line $b = c$ might emerge if this variable also feeds back on itself.

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