Modeling procedural rationality, power relations, and stock-flow consistency: a general constrained dynamics approach

Oliver Richters[a], Erhard Glötzl[b]

[a] International Economics, Department of Economics, Carl von Ossietzky University of Oldenburg, Ammerländer Heerstraße 114–8, 26129 Oldenburg (Oldb), Germany, oliver.richters@uni-oldenburg.de;
[b] Institute for the Comprehensive Analysis of the Economy, Johannes Kepler University Linz, Austria.

submitted to First Behavioral Macroeconomics Workshop: New Approaches to Macro-Financial Instability and Inequality, Bamberg. Version: March 30, 2018.

Keywords: Economic models; Disequilibrium Dynamics; Stock-Flow Consistent Models; Post Keynesian; Constrained Dynamics; Lagrangian Mechanics. **JEL:** B22; C30; C63; E12; E16; E17.

Abstract: In macroeconomic monetary Stock-Flow Consistent (SFC) models, each accounting identity reduces the number of behavioral functions to avoid an overdetermined system. We relax this restriction using a differential-algebraic equation framework of constrained dynamics. Agents exert (ex-ante) forces on the variables according to their desire and their economic power to assert their interest. In analogy to Lagrangian mechanics, system constraints generate additional constraint forces that lead to the ex-post dynamics. We explain the procedure on the basis of a simple SFC model and reveal its implicit assumptions about power relations. Its behavioral assumptions are approximated by agents trying to gradually improve their utility, formalizing a procedural rationality.

1. Introduction

Many general equilibrium models incorporate the notion of neutrality of money, meaning that in the long run, real variables are not influenced by the quantity of money (Patinkin, 2008). These models tend to either neglect short-run adjustment processes completely or treat them as minor nominal rigidities. Differently, three strands of literature have emphasized the need to model the dynamics of financial stocks and flows explicitly. First, the monetary theory of production in the tradition of Keynes (1930, 1973) tries to integrate money and financial assets into macroeconomic models (Godley and Lavoie, 2012; Graziani, 2003). Second, Tobin (1982) and his successors showed interest in portfolio allocation among different financial assets. Third, models based on interconnected balance sheets were praised by Bezemer (2010), arguing that the recent financial crisis was predicted nearly exclusively by economists who deployed implicit or explicit macro-accounting frameworks.

Financial stocks and flows are related by constraints that "remove many degrees of freedom" (Taylor, 2004, p. 2): Modelers have to respect the "interdependence of asset markets enforced by balance-sheet relations" (Meyer, 1975, p. 65). Every financial asset corresponds to a liability held by another agent, every financial flow comes from somewhere and goes somewhere and leads to a change in stocks (Nikiforos and Zezza, 2017, p. 4). Godley and Cripps (1983, p. 18) interpreted this as "a fundamental law of macroeconomics analogous to the principle of conservation of energy in physics". Accordingly, one could call it the 'first law of financial economics'. The model's structure has to respect this stock-flow consistency to avoid "black holes" (Godley, 1996, p. 7) or "pitfalls" (Brainard and Tobin, 1968). Mathematically, the constraints imply that variables cannot be varied independently, and models have to meet the challenge to describe dynamics under constraints.

Multi-period structural dynamical macroeconomics models that respect accounting identities and the interconnectedness of balance sheets have become known as Stock-Flow Consistent (SFC) models (e. g. Caverzasi and Godin, 2015; Dos Santos and Zezza, 2008; Lavoie and Godley, 2001; Le Heron and Mouakil, 2008). Most SFC models are formulated in discrete time, "responding to the way the data are presented" (Taylor, 2008). To specify the time evolution of a model that consists of I variables, exactly I independent equations are needed to avoid an over- or underdetermined system. Because K structural equations such as accounting identities or system constraints have to be satisfied, it is impossible to specify behavioral functions for each of the I variables. The system of equations has to be determined by choosing $I - K$ behavioral equations. They specify the interest and ability of different economic agents to influence key economic variables and describe (often implicit) assumptions about *agency* and *economic power*. From a modeler's perspective, the choice of $I - K$ behavioral equations is permissible if and only if exactly K variables are unaffected by behavioral considerations. This is a serious restriction, and we present a way out of this dilemma.

The framework of General Constrained Dynamics (GCD) allows to specify behavioral forces for *each* of the I variable, still treating the K constraints consistently (section 2). K Lagrangian multipliers are added to make the otherwise overdetermined system of differential-algebraic equations solvable. The constraints generate constraint forces which account for the difference between the planned (ex-ante) and the actual (ex-post) dynamics, and guarantee stock-flow consistency. The behavior is formulated by agents that apply forces to change variables according to their desire and their economic power to assert their interest. Often, economic forces can be formalized as agents trying to increase their utility with a gradient climbing approach, formalizing the "procedural optimizing" discussed by Munier et al. (1999, p. 244) as a possible modeling strategy for bounded rationality. This makes implicit assumptions about agency and power relations in conventional SFC models explicit. It allows to study different power relations and a variety of behavioral forces within a unified mathematical framework in continuous time.

The aim of this paper is to introduce the GCD framework and its methodology on the basis of a comparison with the most simple discrete-time SFC model: the 'SIM' model (for 'simple') from the textbook by Godley and Lavoie (2012, ch. 3), the "main reference work on the methodology" according to Caverzasi and Godin (2015). In section 3, we first construct a continuous-time SFC model in analogy to the textbook model and prove their equivalence analytically. Second, we formulate a similarly structured GCD model and study different specifications of behavioral functions: Convergent or oscillating behavior can be obtained dependent on the relations of governing forces, economic power, and economic constraint forces. Specific power factors make this model mathematical identical to model SIM, revealing implicit assumptions about power relations of the original model. The SIM model can be approximated with agents that gradually try to improve their utility, formalizing a procedural rationality. Section 4 discusses advantages and caveats of our approach and analysis, and concludes.

2. Lagrangian constrained dynamics of financial models

2.1. Lagrangian Constrained Dynamics

Adapting ideas from classical mechanics to economics is strongly associated with General Equilibrium theory (McLure and Samuels, 2001; Mirowski, 1989). Glötzl et al. (2017) present the General Constrained Dynamics (GCD) framework that extends these analogies from constrained optimization to constrained dynamics, based on the concept of Lagrangian mechanics (Flannery, 2011; Lagrange, 1788). We summarize their economic modeling framework to describe out-of-equilibrium dynamics constrained by accounting identities or other system constraints.

The model economy is described by J agents and I variables x_i, corresponding to prices or interest rates, and stocks and flows of commodities and financial liabilities. The bases of the GCD approach are the concepts of *economic forces* and *economic power*. Based on an analogy to mechanical laws, the dynamics of the economic model are the result of agents wanting to change the state of the model economy, represented by different forces f_i^j.

$$\dot{x}_i = \sum_{j=1}^{J} \mu_i^j f_i^j(x). \tag{1}$$

The μ_i^j are called 'power factors' because they represent the *ability* of a specific force f_i^j to change the state of the economy. The total impact of an agent j on the variable x_i is the product of economic force and power $\mu_i^j f_i^j$, i. e. the product of *desire* and *ability* of agent j to influence an economic variable x_i.

2.2. Gradient climbing as procedural rationality

In many cases, the forces can be specified as gradients of utility functions. If an increase of a certain variable x_i leads to a high increase in utility U^j, the agent j tries to augment this variable over time by applying a positive force f_i^j:

$$f_i^j(x) = \frac{\partial U^j(x)}{\partial x_i}. \tag{2}$$

Utility functions in economic models are strongly linked to the concept of 'perfect rationality' implemented as optimization under constraint. Differently, we assume that agents try to gradually improve their utility. Leijonhufvud (2006, p. 31) associates such an adaptive, "gradient climbing" approach with the concept of "procedural rationality" Introducing SFC models, Godley and Lavoie (2012, pp. 11–6) state their "defiance of the putative maximization of utility by individual agents" and prefer such a procedural rationality. In section 3.5, we approximate their model with forces f_i^j given by the gradients of utility functions $U^j(x_i)$, and justify their claim.

2.3. Constraint forces in economic models

The K constraints Z^k are additional conditions which variables have to fulfill, such as accounting identities. In general, they can be written as:

$$0 = Z^k(x, \dot{x}). \tag{3}$$

If one behavioral equation for each variable is specified in a SFC model that contains I variables, the system is over-determined with $K+I$ equations. The Lagrange closure from classical mechanics introduces K additional variables (Lagrangian multipliers λ^k) and additional constraint forces to solve the system while respecting the constraints.

Glötzl et al. (2017) show how constraints add constraint forces to the equations of motion, guaranteeing that identities hold without the need to explicitly define *a priori* which variables determine others. The constraint forces can be modeled in analogy to classical mechanics by Lagrange multipliers λ^k and the gradient of Z^k. Altogether, *ex-ante* forces f_i^j applied by all agents and constraint forces z_i^k create the *ex-post* dynamics:

$$\dot{x}_i = \sum_{j=1}^{J} \mu_i^j f_i^j(x) + \sum_{k=1}^{K} z_i^k(x, \dot{x}), \tag{4}$$

$$0 = Z^k(x, \dot{x}). \tag{5}$$

Analogously to the constraint forces in physics as settled by Flannery (2011), z_i^k can be calculated as

$$z_i^k(x, \dot{x}) = \lambda^k \frac{\partial Z^k}{\partial x_i}, \tag{6}$$

if $\partial Z^k / \partial \dot{x}_i \equiv 0$ ('holonomic' constraints) and as

$$z_i^k(x, \dot{x}) = \lambda^k \frac{\partial Z^k}{\partial \dot{x}_i} \tag{7}$$

for 'non-holonomic' constraints that depend on $\dot{x}_i$. Eq. (4–7) build a system of differential-algebraic equations which can be solved numerically for $x(t)$ and $\dot{x}(t)$.

3. Formulating a Stock-Flow Consistent model in the General Constrained Dynamics framework

We explain the procedure to set up a monetary General Constrained Dynamics model on the basis of the 'simple' textbook post-Keynesian Stock-Flow Consistent model by Godley and Lavoie (2012, ch. 3) called SIM (here: SIM-GL-discrete). It consists of $I = 6$ variables: two stocks (money held by households M^h which is equal to money emitted by the government

M^g)[1] and four flows (total production Y, taxes T, government expenditures G, consumption expenditure C). Y is assumed to be identical to the wage bill, given by the product of hours worked N and the wage w. For sake of simplicity, we set $w = 1$ and therefore $N = Y$, and omit w and N in our description. The original model structure is depicted in figure 1. In the following, discrete time is indicated by indices $Y_{(t)}$, while we use $Y(t)$ for continuous time. In this section, we present the original SIM model, a continuous-time version, and two different specifications of a GCD model.

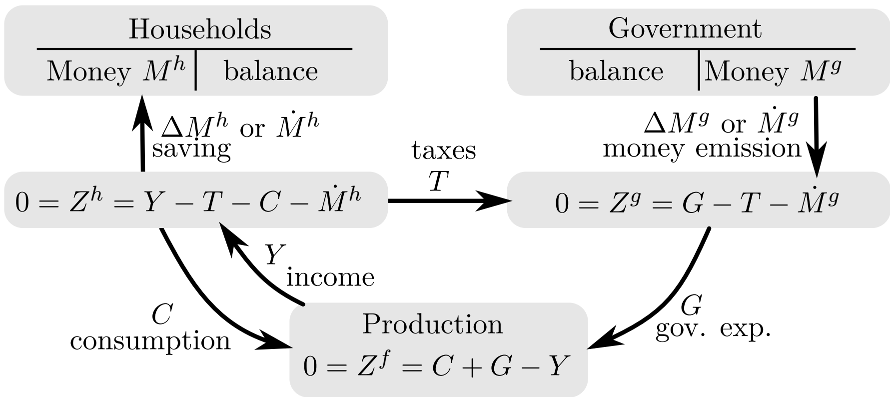

Figure 1: Structure of model SIM (Godley and Lavoie, 2012, ch. 3). In the discrete model, $\dot{M}^h$ and $\dot{M}^g$ correspond to ΔM^h and ΔM^g. The balance sheets and the monetary flows connecting them are depicted.

3.1. Model SIM-GL-discrete: The discrete time SIM model by Godley & Lavoie

The structure of the model is given by $K = 3$ budget constraints for households h, government g, and firms f:

$$0 = Y_{(t)} - T_{(t)} - C_{(t)} - \Delta M^h \qquad \text{with } \Delta M^h = M^h_{(t)} - M^h_{(t-1)}, \tag{8}$$

$$0 = G_{(t)} - T_{(t)} - \Delta M^g \qquad \text{with } \Delta M^g = M^g_{(t)} - M^g_{(t-1)}, \tag{9}$$

$$0 = G_{(t)} + C_{(t)} - Y_{(t)}. \tag{10}$$

[1] Godley and Lavoie (2012) use H for high powered money, we prefer M to avoid confusion with the household.

An initial condition guarantees that public sector debt is equal to private wealth:

$$M^g_{(0)} = M^h_{(0)}. \tag{11}$$

The setup of the model contains $I = 6$ variables and $K = 3$ constraints. $I - K = 3$ behavioral difference equations can be chosen to determine the time evolution. Taxes T are levied as a constant fraction θ of income Y, while government expenditure is exogenously set to the target value $G^\top$. Consumption C is composed of a fraction α_1 of disposable income $Y - T$ and a share α_2 of accumulated wealth at the end of the former period $M^h_{(t-1)}$:

$$T_{(t)} = \theta Y_{(t)}, \tag{12}$$

$$G_{(t)} = G^\top, \tag{13}$$

$$C_{(t)} = \alpha_1 \left[Y_{(t)} - T_{(t)}\right] + \alpha_2 M^h_{(t-1)}. \tag{14}$$

Specifying three behavioral functions (for T, G, and C), the system of equations is fully determined. Godley and Lavoie (2012, p. 63) omit any behavioral influence on hours worked, and assume that there are "no supply constraints of any kind", in particular, workers are eager to work as much as demanded. While they justify this assumption by theoretical arguments, it is crucial to point out that the mathematical structure of conventional SFC models enforces such choices, because the constraints limit the number of possible behavioral equations.

According to the stability analysis performed by Richters and Siemoneit (2017, p. 117), the time evolution of $M = M^h = M^g$ is given by:

$$M_{(t)} - M_{(t-1)} = \frac{-\alpha_2 \theta M_{(t-1)} + (1 - \alpha_1)(1 - \theta)G^\top}{1 - \alpha_1(1 - \theta)}. \tag{15}$$

3.2. Model SIM-GL-continuous: The adaptation of model SIM to continuous time

By passing to the limit $\Delta t \to 0$, a discrete model is transformed into a continuous model, replacing differences such as $M_{(t)} - M_{(t-1)}$ by $\dot{M}(t)$. Two of the budget constraints are transformed into differential equations, the third constraint remains a flow relation:

$$Z^h = 0 = Y(t) - T(t) - C(t) - \dot{M}^h(t), \tag{16}$$

$$Z^g = 0 = G(t) - T(t) - \dot{M}^g(t), \tag{17}$$

$$Z^f = 0 = C(t) + G(t) - Y(t). \tag{18}$$

The initial condition and the $I - K = 3$ algebraic equations which describe the behavior of the agents remain unchanged. $M^h_{(t-1)}$ is replaced by $M^h(t)$, because the time step becomes infinitely small:

$$M^g(0) = M^h(0), \tag{19}$$

$$T(t) = \theta Y(t), \tag{20}$$

$$G(t) = G^\top, \tag{21}$$

$$C(t) = \alpha_1 \left[Y(t) - T(t)\right] + \alpha_2 M^h(t). \tag{22}$$

Simplify the system of equations (16–22) with $M = M^h = M^g$ to:

$$\dot{M}(t) = \frac{-\alpha_2 \theta M(t) + (1-\alpha_1)(1-\theta)G^\top}{1 - \alpha_1(1-\theta)}. \tag{23}$$

The difference equation (15) and the differential equation (23) have identical analytical solutions. $M(t)$ converges exponentially to a stationary state M^*:

$$M(t) = M^* + \left[M(0) - M^*\right] \cdot \exp\left(\frac{-\alpha_2\theta}{1-\alpha_1(1-\theta)} t\right), \tag{24}$$

$$M^* = \frac{(1-\alpha_1)(1-\theta)G^\top}{\alpha_2\theta}. \tag{25}$$

The dynamics of all other variables, derived from equation (24), can be compared in the left plots of figure 2. The two models are equivalent as intended.

3.3. Model SIM-GCD: the model setup

We integrate the SIM model into the GCD formalism described in section 2. The model consists of $J = 3$ agents: the government g, the households h and the firm f. The $I = 6$ variables and $K = 3$ constraints are identical to the continuous version (equations 16–18). In its most general form, the system of differential-algebraic equations can be derived from Eq. (4–5). For each variable i, every agent j creates a force f_i^j with corresponding power factors μ_i^j, and the constraints Z^k lead to constraint forces proportional to λ^k. The framework allows to specify behavioral forces for each variable. As an example, $\mu_C^f f_C^f$ in Eq. (26) corresponds to the power and ability of firms to influence household's consumption. In many applications, several of these terms will be set to 0.

$$\dot{C}(t) = \mu_C^h f_C^h + \mu_C^f f_C^f + \mu_C^g f_C^g - \lambda^h(t) + \lambda^f(t), \tag{26}$$

$$\dot{Y}(t) = \mu_Y^h f_Y^h + \mu_Y^f f_Y^f + \mu_Y^g f_Y^g + \lambda^h(t) - \lambda^f(t), \tag{27}$$

$$\dot{M}^h(t) = \mu_{Mh}^h f_{Mh}^h + \mu_{Mh}^f f_{Mh}^f + \mu_{Mh}^g f_{Mh}^g - \lambda^h(t), \tag{28}$$

$$\dot{M}^g(t) = \mu_{Mg}^h f_{Mg}^h + \mu_{Mg}^f f_{Mg}^f + \mu_{Mg}^g f_{Mg}^g - \lambda^g(t), \tag{29}$$

$$\dot{G}(t) = \mu_G^h f_G^h + \mu_G^f f_G^f + \mu_G^g f_G^g + \lambda^g(t) + \lambda^f(t), \tag{30}$$

$$\dot{T}(t) = \mu_T^h f_T^h + \mu_T^f f_T^f + \mu_T^g f_T^g - \mu_T^h T(t) - \lambda^h(t) - \lambda^g(t), \tag{31}$$

$$0 = Z^h = Y(t) - T(t) - C(t) - \dot{M}^h(t), \tag{32}$$

$$0 = Z^g = G(t) - T(t) - \dot{M}^g(t), \tag{33}$$

$$0 = Z^f = C(t) + G(t) - Y(t), \tag{34}$$

$$0 = M^g(0) - M^h(0). \tag{35}$$

From Eq. (32–35), it follows that $M^h(t) = M^g(t) \; \forall \; t$.

In the following, we illustrate a variety of possible choices for the forces f_i^j, specified either directly or as gradients of some utility function. First, we formulate a model that is equivalent to SIM-GL-continuous for specific power parameters, revealing the assumptions about power by Godley and Lavoie (2012). Second, we implement households forces as gradients of a utility function and show that it approximates SIM-GL-continuous for specific utility and power parameters.

In both cases, we describe the government by the following rules: If taxes are below (above) the targeted fraction θ of income Y and government expenditures G below (above) its target $G^{\top}$, the government tries to increase (decrease) them. This leads to the forces

$$f_T^g = \theta Y(t) - T(t) = \frac{\partial U^g}{\partial T(t)}, \tag{36}$$

$$f_G^g = G^{\top} - G(t) = \frac{\partial U^g}{\partial G(t)}. \tag{37}$$

This is equivalent to the gradient climbing approach with the utility function

$$U^g = -0.5 \left[G^{\top} - G(t)\right]^2 - 0.5 \left[\theta Y(t) - T(t)\right]^2. \tag{38}$$

3.4. Model GCD-forces

For households, we assume that households target a consumption given by a fraction of income and wealth as in the SIM model. Thus if consumption $C(t)$ is below the target, they try to raise C by applying a positive force:

$$f_C^h = \alpha_1 \left[Y(t) - T(t)\right] + \alpha_2 M^h(t) - C(t). \tag{39}$$

Households target $N^{\top}$ working hours. Because of the simplification that the wage rate is given by $w = 1$, this corresponds to a targeted production $Y = wN$ of $N^{\top}$. Therefore, households apply the economic force f_Y^h to change Y:

$$f_Y^h = N^{\top} - Y(t). \tag{40}$$

Households target a stock of money $M^{h\top}$, which yields an economic force they apply to change the stock of money:

$$f^h_M = M^{h\top} - M^h(t). \tag{41}$$

We assume that households try to reduce taxes, for instance by tax evasion. Therefore, we set:

$$f^h_T = -T. \tag{42}$$

We assume that firms try to increase government expenditures and households consumption (for example by lobbying or advertising):

$$f^f_G = G(t), \tag{43}$$

$$f^f_C = C(t). \tag{44}$$

Applying these assumptions to Eq. (4–5), the system of differential-algebraic equations is given by:

$$\dot{C}(t) = \mu^h_C \left[\alpha_1(Y(t) - T(t)) + \alpha_2 M^h(t) - C(t)\right] + \mu^f_C C(t) - \lambda^h(t) + \lambda^f(t), \tag{45}$$

$$\dot{Y}(t) = \mu^h_Y \left[N^\top - Y(t)\right] + \lambda^h(t) - \lambda^f(t), \tag{46}$$

$$\dot{M}^h(t) = \mu^h_M \left[M^{h\top} - M^h(t)\right] - \lambda^h(t), \tag{47}$$

$$\dot{M}^g(t) = \mu^g_M \left[M^\top_g - M^g(t)\right] - \lambda^g(t), \tag{48}$$

$$\dot{G}(t) = \mu^g_G \left[G^\top - G(t)\right] + \mu^f_G G(t) + \lambda^g(t) + \lambda^f(t), \tag{49}$$

$$\dot{T}(t) = \mu^g_T \left[\theta Y(t) - T(t)\right] - \mu^h_T T(t) - \lambda^h(t) - \lambda^g(t), \tag{50}$$

$$0 = Z^h = Y(t) - T(t) - C(t) - \dot{M}^h(t), \tag{51}$$

$$0 = Z^g = G(t) - T(t) - \dot{M}^g(t), \tag{52}$$

$$0 = Z^f = C(t) + G(t) - Y(t), \tag{53}$$

$$0 = M^g(0) - M^h(0). \tag{54}$$

This setup shows that different targets that may even by incompatible ex-ante can be specified, because the constraint forces guarantee ex-post consistency.

For model SIM-GCD-forces to become identical to SIM-GL-continuous, we have to choose the following power factors:[2]

$$\infty = \mu^h_C = \mu^g_G = \mu^g_T, \tag{55}$$

$$0 = \mu^f_C = \mu^h_Y = \mu^h_M = \mu^g_M = \mu^f_G = \mu^h_T. \tag{56}$$

This implies that the government controls expenditure and taxes perfectly (and therefore the change in debt), but chooses not to influence its *level* of debt. Firms do neither influence consumption and government expenditures, nor taxes. Households influence their consumption expenditure, but do not actively influence working hours and their level of money holding. This reveals the implicit assumptions about power relations within the model by Godley and Lavoie (2012), namely the ability of agents' to influence certain variables. The resulting system of equation is identical to the model SIM-GL-continuous in section 3.2. Figure 2b shows the model identity numerically, approximating $\mu = \infty$ by $\mu = 20$.

These power factors seem extreme and one could blame us for choosing an oversimplified original model. In fact, they reflect the general challenge of all conventional SFC models described in the introduction: the existence of constraints reduces the number of behavioral functions, thus one has to choose one-sided influence on some of the interconnected variables in order to make the model solvable. The General Constrained Dynamics approach avoids this kind of technical restrictions. As an example, firms can be assumed to influence government expenditures positively ($\mu^f_G > 0$). The model converges to a stationary state with higher government expenditures that leads (via multipliers) to higher households' consumption (Figure 2c). If firms have the power to raise consumption expenditure ($\mu^f_C > 0$), the dynamics starts to

[2] Divide Eq. (45) by μ^h_C, (49) by μ^g_G, and (50) by μ^g_T. Taking the limit $\rightarrow \infty$, we get the algebraic equations (20), (21), and (22). Setting $\mu^h_Y = 0$ in Eq. (46), $\dot{Y}(t)$ is determined solely by constraint forces and Eq. (46) is linearly dependent on the constraints and therefore can be omitted ('drop closure'; Glötzl, 2015, pp. 33–8). For the same reason, equations (47) and (48) can be omitted. Note that mathematically, the SIM model could also be derived starting from a different system of equations leading to a different set of power factors, but our choice reflects the behavioral assumptions by Godley and Lavoie (2012).

oscillate and yield implausible results, because the government debt M becomes negative and households are indebted to the state. Finally, the model converges to the same stationary consumption C and government expenditure G as model SIM-GL-continuous but the stationary state of the money stocks are different (Figure 2d).

3.5. Model GCD-utility, gradient climbing by households

In section 2.2, we argued that an adaptive rationality can be described by households climbing the gradient of some well-defined utility function. The SIM model can be approximated by assuming that households get positive utility from consumption C and holding liquidity M^h, similar to the 'money in the utility function' approach (Feenstra, 1986; Sidrauski, 1967). We assume a Cobb–Douglas utility function for money and consumption, and an additional term reflecting the disutility of work (as utility of leisure). The scaling is chosen to approximate the other models.

$$U^h = 40C^{a_C}(M^h)^{a_M} + (200 - Y)^{0.5}. \tag{57}$$

According to the power relations revealed in section 3.4, we assume that government has perfect control over spending and taxation ($\mu_G^g = \mu_T^g = \infty$), thus we replace $G(t)$ by $G^\top$ and $T(t)$ by $\theta Y(t)$. The GCD equations are:

$$\dot{C}(t) = \mu_C^h \frac{\partial U^h}{\partial C} - \lambda^h(t) + \lambda^f(t), \tag{58}$$

$$\dot{Y}(t) = \mu_Y^h \frac{\partial U^h}{\partial Y} + (1-\theta)\lambda^h(t) - \theta\lambda^g(t) - \lambda^f(t), \tag{59}$$

$$\dot{M}^h(t) = \mu_M^h \frac{\partial U^h}{\partial M^h} - \lambda^h(t), \tag{60}$$

$$\dot{M}^g(t) = -\lambda^g(t), \tag{61}$$

$$0 = Z^h = (1-\theta)Y(t) - C(t) - \dot{M}^h(t), \tag{62}$$

$$0 = Z^g = G^\top - \theta Y(t) - \dot{M}^g(t), \tag{63}$$

$$0 = Z^f = C(t) + G^\top - Y(t), \tag{64}$$

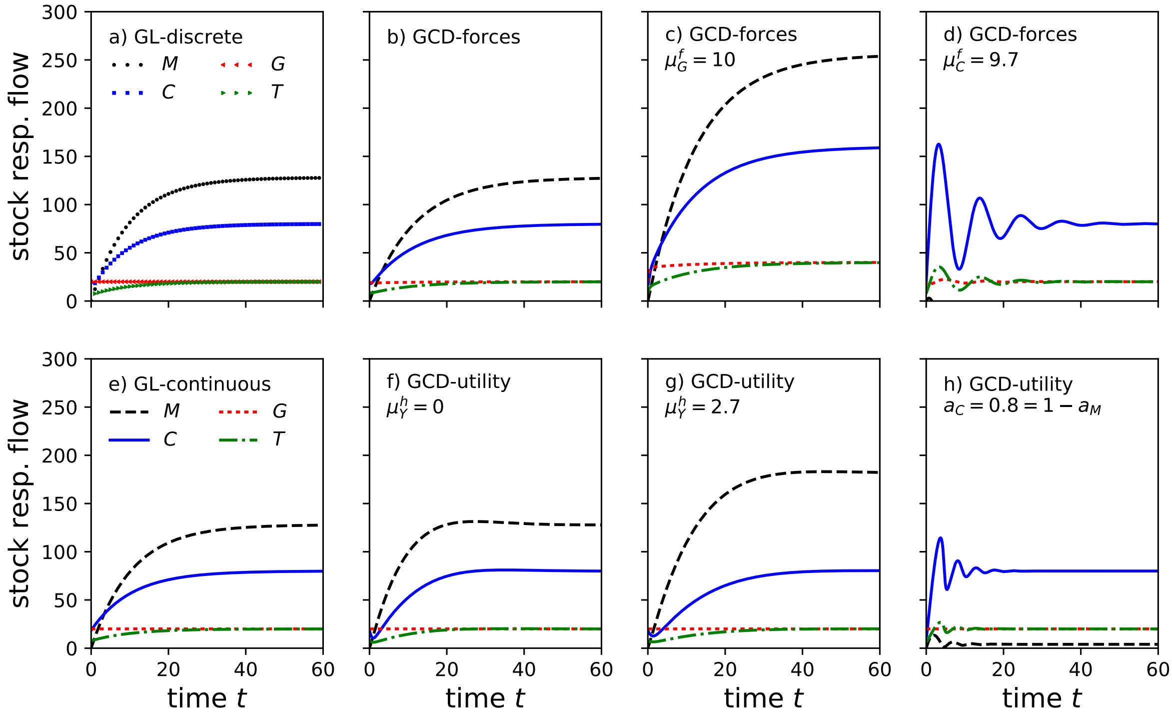

Figure 2: **Plot a:** Original model SIM-GL-discrete following Godley and Lavoie (2012, ch. 3). **Plot e**: continuous adaptation SIM-GL-continuous from section 3.2. Parameters: $\alpha_1 = 0.6$, $\alpha_2 = 0.25$, $\theta = 0.2$. Initial conditions: $M^h(0) = M^g(0) = 0$, $G(0) = G^\top = 20$, $C(0) = (\alpha_1(1-\theta)G(0) + \alpha_2 M(0))/(1-\alpha_1(1-\theta))$, $T(0) = \theta Y(0)$. **Plot b, c, d:** Model SIM-GCD-forces from section 3.3. Parameters: $M^{h\top} = 80$, $M^{g\top} = 80$, $N^\top = 100$, $G^\top = 20$, $\alpha_1 = 0.6$; $\alpha_2 = 0.25$, $\theta = 0.2$. Power factors: $\mu_C^h = 20$, $\mu_C^f = 0$, $\mu_Y^h = 0$, $\mu_M^h = 0$, $\mu_M^g = 0$, $\mu_T^g = 20$, $\mu_T^h = 0$. $\mu = 20$ serves as an approximation of ∞. In **plot b**, model SIM-continuous is reproduced with high accordance. In **plot c**, firms influence government expenditures via $\mu_G^f = 10$. In **plot d**, firm influence consumption decision via $\mu_C^f = 9.7$. **Plot f, g, h:** model SIM-GCD-utility with gradient seeking behavioral functions described in section 3.5. Utility function U^h according to Eq. (57), $N_{max} = 200$, $a_M = (1-\alpha_1)/(1-\alpha_1+\alpha_2\theta) = 8/9$, $a_C = \alpha_2\theta/(1-\alpha_1+\alpha_2\theta) = 1/9$. Power factors: $\mu_C^h = 2$, $\mu_M^h = 2$, $\mu_G^g = 2$, $\mu_T^g = 2$. In plot f, households have $\mu_Y^h = 0$ influence on hours worked and the model converges to the stationary state of model SIM-GL-continuous as desired. In plot g, the model converges to a higher stock of money, because the power of households $\mu_Y^h = 2.7$ to reduce working hours in accordance with the disutility of work reduces households' consumption. In plot h, the exponents of the utility function are adapted, yielding oscillatory behavior.

$$0 = M^h(0) - M^g(0). \tag{65}$$

As Godley and Lavoie (2012, p. 63) assume that households do not influence the hours worked, we set $\mu_Y^h = 0$, and can simplify the system of equations to:

$$\dot{C}(t) = \mu_C^h \frac{\partial U^h}{\partial C(t)} - \theta \mu_M^h \frac{\partial U^h}{\partial M^h(t)} + \theta(1-\theta)G(t) - \theta^2 C(t), \tag{66}$$

$$\dot{M}^h(t) = (1-\theta)G(t) - \theta C(t). \tag{67}$$

A stationary state is reached if

$$(1-\theta)G^\top = \theta C, \tag{68}$$

$$a_C M^h = \theta a_M C. \tag{69}$$

The numerical dynamics are depicted in figure 2. Plot f shows convergence to a stationary state identical to models SIM. Their model can be approximated by the gradient seeking of a household getting utility from consumption and money holding, but powerless to influence the hours worked. If households have the power to influence working hours ($\mu_Y^h > 0$), the household reduces its consumption slightly, which leads to a higher stock of savings in the stationary state (figure 2g). In plot h, the exponent a_C in the household's utility is changed from 1/9 to 4/5. The system shows an oscillating convergence. It can be verified with a stability analysis of the system (see Appendix A) that this is the case for $a_C > 0.2045$ (and therefore $a_M = 1 - a_C < 0.7955$). The General Constrained Dynamics framework allows to model a demand-driven model without neglecting an influence on supply variables.

4. Discussion and Conclusion

Using the example of the textbook model SIM (Godley and Lavoie, 2012), we showed that Stock-Flow Consistent models can be understood as special cases of General Constrained Dynamic models. The GCD framework showed several advantages over conventional SFC models:

A behavioral equation can be assigned to each of the I variables, instead of a (rather arbitrary) subset of $I-K$ behavioral functions combined with K constraints. The choice of the behavioral functions is a decision on the ability of different economic agents to influence certain variables and therefore reflects economic power. The assumptions about power relations of the SIM model could be revealed and 'mixed' power relations with joint influence of agents on direction and adaptation speed of specific variables can be implemented.

Godley and Lavoie (2012, p. 16) claim that their models are based on 'procedural rationality'. We showed how their behavioral functions can be rewritten using utility functions dependent on consumption and holding of liquidity, if utility maximization is replaced by gradient climbing. This provides a sound foundation for their claim and can help to make the assumptions about agents' behavior more explicit. Nevertheless, behavioral forces different from rationality can be studied, or both may coexist within a single model. The formalization of power may further be applied to questions of political economy.

Our example is limited to the most simple Stock-Flow Consistent model. The method can be extended to more complicated SFC models: Their structure consists of a bigger number of sectors, more complex balance sheets, multiple financial and real assets, creating a set of constraints. This structure is complemented by agency, that may include markup-pricing, slow adaptation to expected sales or animal spirits, liquidity preference etc. The economic forces can be specified either by agents trying to improve a well-defined utility function, without the need for an instantaneous optimization, or by any other specification depending on the modeler's needs. Ex-post Stock-Flow Consistency is guaranteed by the mathematical framework, even if these behavioral forces are incompatible ex-ante.

The framework of General Constrained Dynamics allows to study out-of-equilibrium dynamics of stocks and flows consistently. GCD models can be formulated based on utility functions and constraints – replacing optimization of utility functions by a gradient climbing approach that formalizes a procedural rationality that may eventually converge to a stationary state. This setup bears resemblance to general equilibrium models, but is able to integrate the dynamics of conventional Stock-Flow Consistent models. This may help to bridge the current gap between different modeling approaches. Adapting utility functions and power factors, the theoretical assumptions and dynamics can be varied in a flexible way.

A. Stability analysis of model GCD-utility from section 3.5

We study the stability and convergence to the stationary state with bifurcation theory (Kuznetsov, 2004). Around the stationary state given by $\theta C = (1-\theta)G$ and $a_C M^h = (1-\theta)(1-a_C)G$, the system given by Eq. (66–67) can be linearized with the following Jacobian matrix:

$$J_{uv} = \frac{\partial f_u}{\partial x_v} = \begin{pmatrix} \partial f_C/\partial C & \partial f_C/\partial M^h \\ \partial f_{Mh}/\partial C & \partial f_{Mh}/\partial M^h \end{pmatrix} \tag{70}$$

$$= \begin{pmatrix} \mu_C^h \frac{\partial^2 U^h}{\partial C^2} - \theta \mu_M^h \frac{\partial^2 U^h}{\partial C \partial M^h} - \theta^2 & \mu_C^h \frac{\partial^2 U^h}{\partial C \partial M^h} - \theta \mu_M^h \frac{\partial^2 U^h}{(\partial M^h)^2} \\ -\theta & 0 \end{pmatrix} \tag{71}$$

with

$$\left.\frac{\partial^2 U^h}{\partial C^2}\right|_{eq} = -40 a_C^{a_C} (1-a_C)^{2-a_C} (1-\theta)^{-1} \theta^{2-a_C} G^{-1}, \tag{72}$$

$$\left.\frac{\partial^2 U^h}{\partial C \partial M^h}\right|_{eq} = 40 a_C^{1+a_C} (1-a_C)^{1-a_C} (1-\theta)^{-1} \theta^{1-a_C} G^{-1}, \tag{73}$$

$$\left.\frac{\partial^2 U^h}{(\partial M^h)^2}\right|_{eq} = -40 a_C^{2+a_C} (1-a_C)^{-a_C} (1-\theta)^{-1} \theta^{-a_C} G^{-1}. \tag{74}$$

With $\mu_C^h = \mu_M^h = \mu = 2$, this yields

$$J_{uv} = \begin{pmatrix} -\frac{40\mu a_C^{a_C} (1-a_C)^{1-a_C} \theta^{2-a_C}}{(1-\theta)G} - \theta^2 & \frac{40\mu a_C^{1+a_C} \theta^{1-a_C}}{(1-a_C)^{a_C} (1-\theta) G} \\ -\theta & 0 \end{pmatrix} \tag{75}$$

In general, a matrix given by $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ has eigenvalues $\lambda = A/2 \pm \sqrt{A^2/4 + BC}$. For $A < 0$ as in this case, the eigenvalues are complex and the system shows oscillatory behavior if and only if $A^2/4 + BC < 0$. This equation can be solved numerically for a_C, yielding $a_C > 0.2045$ as a condition for oscillations.

Acknowledgements

We thank Florentin Glötzl, Hans-Michael Trautwein, and Johannes Schmidt for their helpful comments. All errors and omissions are our own. OR acknowledges financial support from Ev. Studienwerk Villigst and Sustainable Money Research Group.

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